GLOBAL SOLUTIONS APPROACHING LINES AT INFINITY TO SECOND ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. This article is concerned with second order nonlinear delay, and especially ordinary, differential equations. By the use of the fixed point technique based on the classical Schauder's theorem, for any given line, sufficient conditions are established in order that there exists at least one global solution which is asymptotic at ∞ to this line. In the special case of ordinary differential equations, via the Banach's Contraction Principle, for any given line, conditions are presented which guarantee that there exists a unique global solution that is asymptotic at ∞ to this line. The application of the results obtained to second order delay, and ordinary, differential equations of Emden-Fowler type is presented, and some examples demonstrating the applicability of the results are given. Finally, some supplementary results are obtained, which provide sufficient conditions for all global solutions belonging to a suitable class to be asymptotic at ∞ to lines.

1. INTRODUCTION

In the asymptotic theory of delay, and especially of ordinary, differential equations, an interesting problem is that of the study of solutions with prescribed asymptotic behavior. This problem has been the subject of many investigations; we restrict ourselves to mention the recent papers [2], [5], [10−20] and [22−26] as well as the older classical articles [8, 9] (for a more extensive bibliography, see [17,18]). It is of special interest to investigate global solutions, i.e. solutions on the whole given interval, with prescribed asymptotic behavior. On this problem there is an extensive bibliography (see, for example, [2], [5], [8, 9], [11-16] and [22−26]; for more references, see [15,17,18]). The present work deals with global solutions that are asymptotic at ∞ to lines for second order delay, and especially ordinary, differential equations. For the basic theory of delay differential equations, the reader is referred to the books [3,4,6].

In [17], the authors considered n-th order ($n > 1$) nonlinear ordinary differential equations and studied solutions that behave asymptotically like polynomials at ∞. More precisely, for each given integer m with $1 \leq m \leq n-1$, sufficient conditions have been presented in order that, for any real polynomial of degree at most m, there exists a solution which is asymptotic at ∞ to this polynomial. Conditions

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have been also given, which are sufficient for every solution to be asymptotic at ∞ to a real polynomial of degree at most $n-1$. The application of the results in [17] to the special case of second order nonlinear ordinary differential equations leads to improved versions of the ones contained in the recent paper by Lipovan [10] and of other related results existing in the literature. Note that the nonlinear term, in the differential equations considered in [17], depends only on the time t and the unknown function x.

In a subsequent paper [18], the first and the third author investigated solutions approaching polynomials at ∞ to the more general case of n-th order ($n > 1$) nonlinear ordinary differential equations, in which the nonlinear term depends on the time t and on $x, x', ..., x^{(N)}$, where x is the unknown function and N is an integer with $0 \leq N \leq n-1$. The results obtained in [18] extend those in [17] concerning the particular case where $N = 0$.

It must be noted that, in [17,18] (as well as in [10]), only nonlinear *ordinary* differential equations are considered and that, in these recent works, solutions defined for all large t, but not always *global*, are investigated.

In the present work, we deal with second order nonlinear delay differential equations, and especially ordinary differential equations, and we study global solutions that are asymptotic at ∞ to lines. More precisely, for any given line $\xi t + \eta$ (ξ and η are real constants), we establish sufficient conditions for the existence of at least one global solution x such that $x(t) = \xi t + \eta + o(1)$ and $x'(t) = \xi + o(1)$, for $t \to \infty$. In the special case of second order nonlinear ordinary differential equations, for any given line $\xi t + \eta$ (with $\xi, \eta \in \mathbf{R}$), we also present conditions guaranteeing the existence and uniqueness of a global solution x satisfying $x(t) = \xi t + \eta + o(1)$ and $x'(t) = \xi + o(1)$, for $t \to \infty$. Moreover, we apply our results to the case of second order delay, and especially ordinary, differential equations of Emden-Fowler type, and we give some examples in order to demonstrate the applicability of the results. Finally, we provide sufficient conditions for every global solution x that belongs to a suitable class to satisfy $x(t) = \xi t + \eta + o(1)$ and $x'(t) = \xi + o(1)$, for $t \to \infty$, where ξ and η are real constants (depending on the solution x).

It is an open question whether the results of the present paper can be extended to the more general case of n-th order ($n > 1$) nonlinear delay, and especially ordinary, differential equations. For such differential equations, it is an open problem to investigate the existence (and the uniqueness, in the special case of ordinary differential equations) of global solutions that are asymptotic at ∞ to real polynomials of degree at most m, where m is a given integer with $1 \leq m \leq n-1$.

Throughout the paper, for any interval I of the real line $\mathbf{R}$ and any subset Ω of $\mathbf{R}$, by $C(I, \Omega)$ we will denote the set of all continuous functions defined on I and having values in Ω. Moreover, r *will be a nonnegative real constant.* Furthermore, if t is a point in the interval $[0, \infty)$ and χ is a continuous real-valued function defined at least on $[t-r, t]$, the notation χ_t will be used for the function in $C([-r, 0], \mathbf{R})$ defined by the formula

$$\chi_t(\tau) = \chi(t+\tau) \quad \text{for } -r \leq \tau \leq 0.$$

We notice that the set $C([-r, 0], \mathbf{R})$ is a Banach space endowed with the usual sup-norm $\|\cdot\|$:

$$\|\psi\| = \max_{-r \leq \tau \leq 0} |\psi(\tau)| \quad \text{for } \psi \in C([-r, 0], \mathbf{R}).$$

Consider the second order nonlinear delay differential equation

(E) $$x''(t) + f(t, x_t, x'(t)) = 0,$$

where f *is a continuous real-valued function defined on the set* $[0,\infty)\times C([-r,0],\mathbf{R})\times \mathbf{R}$.

Consider also, in particular, the second order nonlinear delay differential equation

(E$_0$) $$x''(t) + f_0(t, x_t) = 0,$$

where f_0 *is a continuous real-valued function defined on the set* $[0,\infty)\times C([-r,0],\mathbf{R})$.

We are interested in solutions of the delay differential equations (E) and (E$_0$) on the *whole* interval $[0,\infty)$. By a *solution on* $[0,\infty)$ of (E) [respectively, of (E$_0$)], we mean a function x in $C([-r,\infty),\mathbf{R})$ which is twice continuously differentiable on the interval $[0,\infty)$ and satisfies (E) [resp., (E$_0$)] for all $t \geq 0$.

Furthermore, let us concentrate on a particular class of delay differential equations. More precisely, let us consider the second order nonlinear delay differential equation

(E′) $$x''(t) + g(t, x(t - T_1(t)), ..., x(t - T_m(t)), x'(t)) = 0$$

and, in particular, the second order nonlinear delay differential equation

(E′$_0$) $$x''(t) + g_0(t, x(t - T_1(t)), ..., x(t - T_m(t))) = 0,$$

where m *is a positive integer,* g *is a continuous real-valued function on* $[0,\infty) \times \mathbf{R}^{m+1}$, g_0 *is a continuous real-valued function on* $[0,\infty)\times\mathbf{R}^m$*, and* T_j $(j = 1, ..., m)$ *are nonnegative continuous real-valued functions on the interval* $[0,\infty)$ *with*

$$\max_{j=1,\dots,m}\sup_{t\geq 0} T_j(t) = r.$$

If the delay differential equation (E) or (E$_0$) is to be equivalent to (E′) or (E′$_0$), respectively, we must define

$$f(t,\psi,z) = g(t, \psi(-T_1(t)), ..., \psi(-T_m(t)), z) \quad \text{for any } (t,\psi,z) \in [0,\infty) \times C([-r,0],\mathbf{R}) \times \mathbf{R}$$

or

$$f_0(t,\psi) = g_0(t, \psi(-T_1(t)), ..., \psi(-T_m(t))) \quad \text{for any } (t,\psi) \in [0,\infty) \times C([-r,0],\mathbf{R}),$$

respectively.

We restrict our attention only to solutions of the delay differential equations (E′) and (E′$_0$) on the *whole* interval $[0,\infty)$. A *solution on* $[0,\infty)$ of (E′) [resp., of (E′$_0$) is a function x in $C([-r,\infty),\mathbf{R})$, which is twice continuously differentiable on the interval $[0,\infty)$ and satisfies (E′) [resp., (E′$_0$)] for all $t \geq 0$.

Now, let us consider the special case of *ordinary* differential equations. That is, consider the second order nonlinear ordinary differential equation

(D) $$x''(t) + h(t, x(t), x'(t)) = 0$$

and, especially, the second order nonlinear ordinary differential equation

(D$_0$) $$x''(t) + h_0(t, x(t)) = 0,$$

where h *is a continuous real-valued function on* $[0,\infty) \times \mathbf{R}^2$*, and* h_0 *is is a continuous real-valued function on* $[0,\infty) \times \mathbf{R}$.

We confine our discussion only to solutions of the differential equations (D) and (D$_0$) on the *whole* interval $[0,\infty)$.

The results of the paper are stated in Section 2, while their proofs are given in Sections 3−5. Section 6 is devoted to the application of the results to second order (delay or, especially, ordinary) differential equations of Emden-Fowler type as well as to some examples demonstrating the applicability of our results. In the last section (Section 7) some supplementary results are given, which can be characterized as a complement of the results of the present work.

2. STATEMENT OF THE RESULTS

Our results in this paper are presented in the form of four theorems (Theorems 1−4) and four Corollaries (Corollaries 1−4). In Theorem 1 (respectively, Theorem 2), for given real constants ξ and η, sufficient conditions are established in order that the delay differential equation (E) [resp., ($\mathrm{E_0}$)] have at least one solution x on the interval $[0,\infty)$ such that $x(t) = \xi t + \eta + o(1)$ and $x'(t) = \xi + o(1)$, for $t \to \infty$. Corollary 1 (resp., Corollary 2) is the application of Theorem 1 (resp., Theorem 2) to the particular case of the delay differential equation (E′) [resp., (E'_0)], while Corollary 3 (resp., Corollary 4) is the specialization of Theorem 1 (resp., Theorem 2) to the ordinary differential equation (D) [resp., ($\mathrm{D_0}$)]. In Theorem 3 (resp., Theorem 4), for given real constants ξ and η, conditions are presented, which are suffficient for the ordinary differential equation (D) [resp., ($\mathrm{D_0}$)] to have exactly one solution x on the interval $[0,\infty)$ such that $x(t) = \xi t + \eta + o(1)$ and $x'(t) = \xi + o(1)$, for $t \to \infty$.

Theorem 1. *Assume that*

$$|f(t,\psi,z)| \le F(t,|\psi|,|z|) \quad \text{for all } (t,\psi,z) \in [0,\infty) \times C([-r,0],\mathbf{R}) \times \mathbf{R}, \tag{2.1}$$

where F is a nonnegative real-valued function defined on $[0,\infty)\times C([-r,0],[0,\infty))\times [0,\infty)$, which satisfies the Continuity Condition:

(C) *$F(t,|\chi_t|,|\chi'(t)|)$ is continuous with respect to t in $[0,\infty)$ for each given function χ in $C([-r,\infty),\mathbf{R})$ which is continuously differentiable on the interval $[0,\infty)$.*

Suppose that:

(B) *For each $t \ge 0$, the function $F(t,\cdot,\cdot)$ is increasing on $C([-r,0],[0,\infty)) \times [0,\infty)$ in the sense that $F(t,\psi,z) \le F(t,\omega,v)$ for any ψ, ω in $C([-r,0],[0,\infty))$ with $\psi \le \omega$ (i.e. $\psi(\tau) \le \omega(\tau)$ for $-r \le \tau \le 0$) and any z, v in $[0,\infty)$ with $z \le v$.*

Let ξ and η be given real constants, and let there exist a real number c with $c > \max\{|\xi|,|\eta|\}$ so that

$$\int_0^\infty tF(t,\gamma_t,c)dt \le c - |\eta| \tag{2.2}$$

and

$$\int_0^\infty F(t,\gamma_t,c)dt \le c - |\xi|, \tag{2.3}$$

where the function γ in $C([-r,\infty),[0,\infty))$ depends on c and is defined by

$$\gamma(t) = \begin{cases} c & \text{for } -r \le t \le 0 \\ c(t+1) & \text{for } t \ge 0. \end{cases} \tag{2.4}$$

Then the delay differential equation (E) *has at least one solution* x *on the interval* $[0,\infty)$ *such that*

$$x(t)=\xi t+\eta+o(1) \quad \text{for } t\to\infty \tag{2.5}$$

and

$$x'(t)=\xi+o(1) \quad \text{for } t\to\infty; \tag{2.6}$$

in addition, this solution x *satisfies*

$$x(t)=x(0) \quad \text{for } -r\le t\le 0, \tag{2.7}$$

$$\xi t+\eta-(c-|\eta|)\le x(t)\le \xi t+\eta+(c-|\eta|) \quad \text{for every } t\ge 0 \tag{2.8}$$

and

$$\xi-(c-|\xi|)\le x'(t)\le \xi+(c-|\xi|) \quad \text{for every } t\ge 0. \tag{2.9}$$

Theorem 2. *Assume that*

$$|f_0(t,\psi)|\le F_0(t,|\psi|) \quad \text{for all } (t,\psi)\in[0,\infty)\times C([-r,0],\mathbf{R}), \tag{2.10}$$

where F_0 *is a nonnegative real-valued function defined on* $[0,\infty)\times C([-r,0],[0,\infty))$, *which satisfies the Continuity Condition:*

(C_0) $F_0(t,|\chi_t|)$ *is continuous with respect to* t *in* $[0,\infty)$ *for each given function* χ *in* $C([-r,\infty),\mathbf{R})$.

Suppose that:

(B_0) *For each* $t\ge 0$, *the function* $F_0(t,\cdot)$ *is increasing on* $C([-r,0],[0,\infty))$ *in the sense that* $F_0(t,\psi)\le F_0(t,\omega)$ *for any* ψ, ω *in* $C([-r,0],[0,\infty))$ *with* $\psi\le\omega$ (*i.e.* $\psi(\tau)\le\omega(\tau)$ *for* $-r\le\tau\le 0$).

Let ξ *and* η *be given real constants, and let there exist a real number* c *with* $c>\max\{|\xi|,|\eta|\}$ *so that*

$$\int_0^\infty tF_0(t,\gamma_t)dt\le c-\max\{|\xi|,|\eta|\}, \tag{2.11}$$

where the function γ *in* $C([-r,\infty),[0,\infty))$ *depends on* c *and is defined by* (2.4). *Then the delay differential equation* (E_0) *has at least one solution* x *on the interval* $[0,\infty)$ *such that* (2.5) *and* (2.6) *hold; in addition, this solution* x *satisfies* (2.7) *and:*

$$\xi t+\eta-(c-\max\{|\xi|,|\eta|\})\le x(t)\le \xi t+\eta+(c-\max\{|\xi|,|\eta|\}) \quad \text{for every } t\ge 0 \tag{2.12}$$

and

$$\xi-\int_0^\infty F_0(s,\gamma_s)ds\le x'(t)\le \xi+\int_0^\infty F_0(s,\gamma_s)ds \quad \text{for every } t\ge 0. \tag{2.13}$$

(*Note that, because of* (2.11), $\int_0^\infty F_0(s,\gamma_s)ds$ *is finite.*)

Corollary 1. *Assume that*

$$|g(t,y_1,\dots,y_m,z)|\le G(t,|y_1|,\dots,|y_m|,|z|) \quad \text{for } (t,y_1,\dots,y_m,z)\in[0,\infty)\times\mathbf{R}^{m+1},$$

where G *is a nonnegative continuous real-valued function on* $[0,\infty)\times[0,\infty)^{m+1}$. *Suppose that:*

(B′) *For each* $t \geq 0$, *the function* $G(t, \cdot, ..., \cdot, \cdot)$ *is increasing on* $[0, \infty)^{m+1}$ *in the sense that* $G(t, y_1, ..., y_m, z) \leq G(t, w_1, ..., w_m, v)$ *for any* $(y_1, ..., y_m, z)$, $(w_1, ..., w_m, v)$ *in* $[0, \infty)^{m+1}$ *with* $y_1 \leq w_1, ..., y_m \leq w_m$, $z \leq v$.

Let ξ *and* η *be given real constants, and let there exist a real number* c *with* $c > \max\{|\xi|, |\eta|\}$ *so that*

$$\int_0^{\infty} tG(t, \rho_1(t), ..., \rho_m(t), c)dt \leq c - |\eta|$$

and

$$\int_0^{\infty} G(t, \rho_1(t), ..., \rho_m(t), c)dt \leq c - |\xi|,$$

where, for each $j \in \{1, ..., m\}$, *the function* ρ_j *in* $C([0, \infty), [0, \infty))$ *depends on* c *and is defined by*

$$\rho_j(t) = \begin{cases} c, & \text{if } 0 \leq t \leq T_j(t) \\ c(t - T_j(t) + 1), & \text{if } t \geq T_j(t). \end{cases} \tag{2.14}$$

Then the delay differential equation (E′) *has at least one solution* x *on the interval* $[0, \infty)$ *such that* (2.5) *and* (2.6) *hold; in addition, this solution* x *satisfies* (2.7), (2.8) *and* (2.9).

Corollary 2. *Assume that*

$$|g_0(t, y_1, ..., y_m)| \leq G_0(t, |y_1|, ..., |y_m|) \quad \text{for } (t, y_1, ..., y_m) \in [0, \infty) \times \mathbf{R}^m,$$

where G_0 *is a nonnegative continuous real-valued function on* $[0, \infty) \times [0, \infty)^m$. *Suppose that:*

(B_0') *For each* $t \geq 0$, *the function* $G_0(t, \cdot, ..., \cdot)$ *is increasing on* $[0, \infty)^m$ *in the sense that* $G_0(t, y_1, ..., y_m) \leq G_0(t, w_1, ..., w_m)$ *for any* $(y_1, ..., y_m)$, $(w_1, ..., w_m)$ *in* $[0, \infty)^m$ *with* $y_1 \leq w_1, ..., y_m \leq w_m$.

Let ξ *and* η *be given real constants, and let there exist a real number* c *with* $c > \max\{|\xi|, |\eta|\}$ *so that*

$$\int_0^{\infty} tG_0(t, \rho_1(t), ..., \rho_m(t))dt \leq c - \max\{|\xi|, |\eta|\}, \tag{2.15}$$

where, for each $j \in \{1, ..., m\}$, *the function* ρ_j *in* $C([0, \infty), [0, \infty))$ *depends on* c *and is defined by* (2.14). *Then the delay differential equation* (E_0') *has at least one solution* x *on the interval* $[0, \infty)$ *such that* (2.5) *and* (2.6) *hold; in addition, this solution* x *satisfies* (2.7), (2.12), *and*

$$\xi - \int_0^{\infty} G_0(s, \rho_1(s), ..., \rho_m(s))ds \leq x'(t) \leq \xi + \int_0^{\infty} G_0(s, \rho_1(s), ..., \rho_m(s))ds \quad \text{for every } t \geq 0.$$

(*Note that, because of* (2.15), $\int_0^{\infty} G_0(s, \rho_1(s), ..., \rho_m(s))ds$ *is finite.*)

Corollary 3. *Assume that*

$$|h(t, y, z)| \leq H(t, |y|, |z|) \quad \text{for all } (t, y, z) \in [0, \infty) \times \mathbf{R}^2, \tag{2.16}$$

where H *is a nonnegative continuous real-valued function on* $[0, \infty) \times [0, \infty)^2$. *Suppose that:*

(A) *For each* $t \geq 0$, *the function* $H(t, \cdot, \cdot)$ *is increasing on* $[0, \infty)^2$ *in the sense that* $H(t, y, z) \leq H(t, w, v)$ *for any* (y, z), (w, v) *in* $[0, \infty)^2$ *with* $y \leq w$, $z \leq v$.

Let ξ and η be given real constants, and let there exist a real number c with $c > \max\{|\xi|, |\eta|\}$ so that

$$\int_0^\infty tH(t, c(t+1), c)dt \leq c - |\eta| \tag{2.17}$$

and

$$\int_0^\infty H(t, c(t+1), c)dt \leq c - |\xi| . \tag{2.18}$$

Then the ordinary differential equation (D) *has at least one solution x on the interval $[0, \infty)$ such that* (2.5) *and* (2.6) *hold; in addition, this solution x satisfies* (2.8) *and* (2.9).

Corollary 4. *Assume that*

$$|h_0(t, y)| \leq H_0(t, |y|) \quad \text{for all } (t, y) \in [0, \infty) \times \mathbf{R}, \tag{2.19}$$

where H_0 is a nonnegative continuous real-valued function on $[0, \infty) \times [0, \infty)$. Suppose that:

(A_0) *For each $t \geq 0$, the function $H_0(t, \cdot)$ is increasing on $[0, \infty)$ in the sense that $H_0(t, y) \leq H_0(t, w)$ for any y, w in $[0, \infty)$ with $y \leq w$.*

Let ξ and η be given real constants, and let there exist a real number c with $c > \max\{|\xi|, |\eta|\}$ so that

$$\int_0^\infty tH_0(t, c(t+1))dt \leq c - \max\{|\xi|, |\eta|\}. \tag{2.20}$$

Then the ordinary differential equation (D_0) *has at least one solution x on the interval $[0, \infty)$ such that* (2.5) *and* (2.6) *hold; in addition, this solution x satisfies* (2.12) *and*

$$\xi - \int_0^\infty H_0(s, c(s+1))ds \leq x'(t) \leq \xi + \int_0^\infty H_0(s, c(s+1))ds \quad \text{for every } t \geq 0. \tag{2.21}$$

(*Note that, because of* (2.20), *$\int_0^\infty H_0(s, c(s+1))ds$ is finite.*)

Theorem 3. *Let the following generalized Lipschitz condition be satisfied:*

$$|h(t, y, z) - h(t, w, v)| \leq L(t) \max\{|y - w|, |z - v|\} \quad \text{for all } (t, y, z), (t, w, v) \text{ in } [0, \infty) \times \mathbf{R}^2, \tag{2.22}$$

where L is a nonnegative continuous real-valued function on the interval $[0, \infty)$ such that

$$\max\left\{\int_0^\infty t(t+1)L(t)dt, \ \int_0^\infty (t+1)L(t)dt\right\} < 1. \tag{2.23}$$

Moreover, assume that (2.16) *holds, where H is a nonnegative continuous real-valued function on $[0, \infty) \times [0, \infty)^2$. Suppose that* (A) *is satisfied.*

Let ξ and η be given real constants, and let there exist a real number c with $c > \max\{|\xi|, |\eta|\}$ so that (2.17) *and* (2.18) *hold. Then the ordinary differential equation* (D) *has exactly one solution x on the interval $[0, \infty)$ with*

$$|x(0)| \leq c \tag{2.24}$$

and

$$|x'(t)| \leq c \quad \textit{for every } t \geq 0, \tag{2.25}$$

such that (2.5) *and* (2.6) *hold; in addition, this unique solution* x *satisfies* (2.8) *and* (2.9).

Theorem 4. *Let the following generalized Lipschitz condition be satisfied:*

$$|h_0(t,y) - h_0(t,w)| \leq L_0(t)\,|y-w| \quad \textit{for all } (t,y),(t,w) \textit{in } [0,\infty)\times \mathbf{R}, \tag{2.26}$$

where L_0 *is a nonnegative continuous real-valued function on the interval* $[0,\infty)$ *such that*

$$\int_0^\infty t(t+1)L_0(t)dt < 1. \tag{2.27}$$

Moreover, assume that (2.19) *holds, where* H_0 *is a nonnegative continuous real-valued function on* $[0,\infty)\times[0,\infty)$. *Suppose that* (A_0) *is satisfied.*

Let ξ *and* η *be given real constants, and let there exist a real number* c *with* $c > \max\{|\xi|, |\eta|\}$ *so that* (2.20) *holds. Then the ordinary differential equation* (D_0) *has exactly one solution* x *on the interval* $[0,\infty)$ *with*

$$|x(t)| \leq c(t+1) \quad \textit{for every } t \geq 0, \tag{2.28}$$

and such that (2.5) *and* (2.6) *hold; in addition, this unique solution* x *satisfies* (2.12) *and* (2.21).

Note: Inequalities (2.24) and (2.25) imply (2.28).

An important remark. (i) In the conclusions of Theorems 1 and 3 and of Corollaries 1 and 3, the solution x satisfies (2.8) and (2.9).

Assume that $\xi > 0$ *and* $\eta > 0$. Then (2.8) and (2.9) are written as

$$\xi t - (c - 2\eta) \leq x(t) \leq \xi t + c \quad \text{for every } t \geq 0 \tag{2.8$'$}$$

and

$$-(c-2\xi) \leq x'(t) \leq c \quad \text{for every } t \geq 0, \tag{2.9$'$}$$

respectively. Furthermore, *in addition to the hypothesis* $c > \xi$ *and* $c > \eta$, *let us suppose that* $c < 2\xi$ *and* $c \leq 2\eta$. *We have thus* $0 < \xi < c < 2\xi$ *and* $0 < \eta < c \leq 2\eta$. Then (2.8′) guarantees that *the solution* x *is positive on the interval* $(0,\infty)$ *and such that* $\lim_{t\to\infty} x(t) = \infty$. Also, from (2.9′) it follows that $x'(t) > 0$ for $t \geq 0$ and so x *is strictly increasing on the interval* $[0,\infty)$.

Analogously, *in the case where* $2\xi < -c < \xi < 0$ *and* $2\eta \leq -c < \eta < 0$, we can see that *the solution* x *is negative on the interval* $(0,\infty)$ *and such that* $\lim_{t\to\infty} x(t) = -\infty$, and that x *is strictly decreasing on* $[0,\infty)$.

(ii) The solution x in the conclusion of Theorem 2 is such that (2.12) and (2.13) are satisfied. (Analogous inequalities are fulfilled for the solution x in the conclusions of Corollaries 2 and 4, and of Theorem 4).

Let ξ *and* η *be positive.* Then (2.12) becomes

$$\xi t - [c - (\eta + \max\{\xi,\eta\})] \leq x(t) \leq \xi t + [c - (-\eta + \max\{\xi,\eta\})] \quad \text{for every } t \geq 0. \tag{2.12$'$}$$

We have assumed that $c > \xi$ and $c > \eta$. *In addition to this assumption, let us suppose that* $\xi > \int_0^\infty F_0(s,\gamma_s)ds$ *and* $c \leq \eta + \max\{\xi,\eta\}$. So, *we have* $0 \leq$

$\int_0^\infty F_0(s,\gamma_s)ds < \xi < c$ and $0 < \eta < c \le \eta + \max\{\xi,\eta\}$. It follows from (2.12′) that *the solution x is positive on the interval $(0,\infty)$ and satisfies $\lim_{t\to\infty} x(t) = \infty$.* Moreover, (2.13) ensures that $x'(t) > 0$ for $t \ge 0$ and consequently *x is strictly increasing on the interval $[0,\infty)$.*

In a similar way, we can conclude that, *if $-c < \xi < -\int_0^\infty F_0(s,\gamma_s)ds \le 0$ and $\eta + \max\{\xi,\eta\} \le -c < \eta < 0$, then the solution x is negative on the interval $(0,\infty)$ with $\lim_{t\to\infty} x(t) = -\infty$, and strictly decreasing on $[0,\infty)$.*

Before closing this section, we must point out the connection between Theorem 1 and Theorem 2. It is obvious that *Theorem 1 concerning the delay differential equation* (E) *is also applicable to the particular case of the delay differential equation* (E_0). It is remarkable that the result obtained by such an application is different from Theorem 2 dealing with the delay differential equation (E_0). As it is evident, the conclusion of Theorem 2 cannot be derived from the conclusion of Theorem 1. What is more, the spaces on which Schauder's theorem is applied in the proofs of these two theorems are different one another. Therefore, the proofs themselves are significantly different. Example 7 at the end of Section 6 illustrates the difference between the conlcusion deduced by Theorem 1 and the conlcusion deduced by Theorem 2.

Analogous remarks can be made for the connection between Corollary 1 and Corollary 2, between Corollary 3 and Corollary 4, and between Theorem 3 and Theorem 4.

3. PROOF OF THEOREM 1

The proof of Theorem 1 is based on the use of the following Schauder's fixed point theorem (see Schauder [21]).

The Schauder theorem. *Let S be a Banach space and X any nonempty convex and closed subset of S. If M is a continuous mapping of X into itself and MX is relatively compact, then the mapping M has at least one fixed point (i.e. there exists an $x \in X$ with $x = Mx$).*

Let $BC([0,\infty),\mathbf{R})$ be the Banach space of all bounded continuous real-valued functions on the interval $[0,\infty)$, endowed with the sup-norm $\|\cdot\|$ defined by

$$\|u\| = \sup_{t\ge 0} |u(t)| \quad \text{for } u \in BC([0,\infty),\mathbf{R}).$$

We need the following compactness criterion for subsets of $BC([0,\infty),\mathbf{R})$, which is a consequence of the well-known Arzelà-Ascoli theorem. This compactness criterion is an adaptation of a lemma due to Avramescu [1].

Compactness criterion. *Let U be an equicontinuous and uniformly bounded subset of the Banach space $BC([0,\infty),\mathbf{R})$. If U is equiconvergent at ∞, it is also relatively compact.*

Note that a set U of real-valued functions defined on the interval $[0,\infty)$ is called *equiconvergent at ∞* if all functions in U are convergent in $\mathbf{R}$ at the point ∞ and, in addition, for each $\epsilon > 0$, there exists $T = T(\epsilon) > 0$ such that, for all functions u in U, it holds $\left|u(t) - \lim_{s\to\infty} u(s)\right| < \epsilon$ for every $t \geq T$.

Throughout the ramainder of this section, by S we will denote the set of all functions in $C([-r,\infty),\mathbf{R})$, which have bounded continuous derivatives on the interval $[0,\infty)$. The set S *is a Banach space endowed with the norm* $\|\cdot\|$ defined as follows

$$\| u \| = \max\left\{ \max_{-r\leq t\leq 0} |u(t)|\,,\ \sup_{t\geq 0} |u'(t)| \right\} \quad \text{for } u \in S.$$

To prove Theorem 1, we first establish the following proposition.

Proposition 1. *Assume that* (2.1) *holds, where F is a nonnegative real-valued function defined on* $[0,\infty)\times C([-r,0],[0,\infty))\times[0,\infty)$, *which satisfies the Continuity Condition* (C). *Suppose that* (B) *is satisfied.*

Let ξ and η be given real constants, and let c be a positive real number such that

$$\int_0^\infty tF(t,\gamma_t,c)dt < \infty, \tag{3.1}$$

where the function γ in $C([-r,\infty),[0,\infty))$ depends on c and is defined by (2.4). *Let also X be the subset of S defined by*

$$X = \{x \in S: \ \| x \| \leq c\}. \tag{3.2}$$

Then the formula

$$(Mx)(t) = \begin{cases} \eta - \int_0^\infty sf(s,x_s,x'(s))ds & \text{for } -r\leq t\leq 0 \\ \xi t + \eta - \int_t^\infty (s-t)f(s,x_s,x'(s))ds & \text{for } t\geq 0 \end{cases} \tag{3.3}$$

makes sense for any function x in X, and this formula defines a continuous mapping M of X into S such that MX is relatively compact.

Proof of Proposition 1. Let x be an arbitrary function in X. From the definition of X, via (3.2), it follows that

$$|x(t)| \leq c \quad \text{for } -r\leq t\leq 0 \tag{3.4}$$

and

$$|x'(t)| \leq c \quad \text{for every } t \geq 0. \tag{3.5}$$

Inequality (3.4) gives, in particular, $|x(0)| \leq c$. So, by using this fact and (3.5), we obtain for $t \geq 0$

$$|x(t)| = \left|x(0) + \int_0^t x'(s)ds\right| \leq |x(0)| + \int_0^t |x'(s)|\,ds \leq c + ct,$$

i.e. we have

$$|x(t)| \leq c(t+1) \quad \text{for every } t\geq 0. \tag{3.6}$$

In view of (2.4), from (3.4) and (3.6) we conclude that

$$|x(t)| \leq \gamma(t) \quad \text{for } t\geq -r$$

and consequently

$$|x_t| \leq \gamma_t \quad \text{for all } t \geq 0. \tag{3.7}$$

By taking into account (3.7) and (3.5) and using the assumption (B), we get

$$F(t, |x_t|, |x'(t)|) \leq F(t, \gamma_t, c) \quad \text{for } t \geq 0.$$

On the other hand, because of (2.1), it holds

$$|f(t, x_t, x'(t))| \leq F(t, |x_t|, |x'(t)|) \quad \text{for } t \geq 0.$$

Thus, we find

$$|f(t, x_t, x'(t))| \leq F(t, \gamma_t, c) \quad \text{for every } t \geq 0. \tag{3.8}$$

Furthermore, by combining (3.1) and (3.8), we have

$$\int_0^\infty t\, |f(t, x_t, x'(t))|\, dt < \infty. \tag{3.9}$$

This, in particular, implies

$$\int_0^\infty |f(t, x_t, x'(t))|\, dt < \infty. \tag{3.10}$$

So, in view of (3.9) and (3.10), it is true that

$$\int_0^\infty t f(t, x_t, x'(t)) dt \quad \text{and} \quad \int_0^\infty f(t, x_t, x'(t)) dt \qquad \text{exist in } \mathbf{R}. \tag{3.11}$$

As (3.11) holds true for all functions x in X, we can immediately see that *the formula* (3.3) *makes sense for any function x in X, and this formula defines a mapping M of X* into $C([-r, \infty), \mathbf{R})$. We will show that M is a mapping of X *into S*, i.e. that $MX \subseteq S$. To this end, let us consider an arbitrary function x in X. Then, by taking into account (3.8), from (3.3) we obtain for $t \geq 0$

$$\begin{aligned} |(Mx)'(t)| &= \left|\xi + \int_t^\infty f(s, x_s, x'(s)) ds\right| \leq |\xi| + \int_t^\infty |f(s, x_s, x'(s))|\, ds \\ &\leq |\xi| + \int_t^\infty F(s, \gamma_s, c) ds \leq |\xi| + \int_0^\infty F(s, \gamma_s, c) ds. \end{aligned}$$

Therefore,

$$|(Mx)'(t)| \leq Q \quad \text{for all } t \geq 0, \tag{3.12}$$

where

$$Q = |\xi| + \int_0^\infty F(s, \gamma_s, c) ds. \tag{3.13}$$

Note that (3.1) guarantees, in particular, that

$$\int_0^\infty F(t, \gamma_t, c) dt < \infty \tag{3.14}$$

and so Q is a nonnegative real constant. Inequality (3.12) means that $(Mx)'$ is always bounded on the interval $[0, \infty)$, and consequently Mx belongs to S. We have thus proved that, for any function x in X, $Mx \in S$, i.e. that $MX \subseteq S$.

Now, we shall prove that MX *is relatively compact.* From (3.3) it follows that, for each $x \in X$, the function Mx is constant on the interval $[-r, 0]$. By taking into account this fact as well as the definition of the norm $\nparallel \cdot \nparallel$, we can easily conclude that it suffices to prove that the set

$$U = \{((Mx) \mid [0, \infty))' : \ x \in X\}$$

is relatively compact in the Banach space $BC([0,\infty),\mathbf{R})$. Each function x in X satisfies (3.12), where the nonnegative real number Q is defined by (3.13) (and it is independent of x). This ensures that U is uniformly bounded. Furthermore, for any function x in X, it follows from (3.3) that

$$|(Mx)'(t)-\xi| = \left|\int_t^\infty f(s,x_s,x'(s))ds\right| \le \int_t^\infty |f(s,x_s,x'(s))|\,ds$$

for all $t \ge 0$, and consequently, by taking into account (3.8), we derive

$$|(Mx)'(t)-\xi| \le \int_t^\infty F(s,\gamma_s,c)ds \quad \text{for every } t\ge 0. \tag{3.15}$$

For any function x in X, (3.15) together with (3.14) imply that

$$\lim_{t\to\infty}(Mx)'(t)=\xi.$$

By using again (3.14) and (3.15), we immediately see that U is equiconvergent at ∞. Now, by (3.8), for any function x in X and every t_1, t_2 with $0 \le t_1 \le t_2$, from (3.3) we obtain

$$\begin{aligned}
&|(Mx)'(t_1)-(Mx)'(t_2)| \\
=\ & \left|\left[\xi+\int_{t_1}^\infty f(s,x_s,x'(s))ds\right]-\left[\xi+\int_{t_2}^\infty f(s,x_s,x'(s))ds\right]\right| \\
=\ & \left|\int_{t_1}^{t_2} f(s,x_s,x'(s))ds\right| \le \int_{t_1}^{t_2}|f(s,x_s,x'(s))|\,ds \\
\le\ & \int_{t_1}^{t_2} F(s,\gamma_s,c)ds.
\end{aligned}$$

Thus, by virtue of (3.14), it is easy to verify that U is equicontinuous. By the given compactness criterion, the set U is relatively compact in $BC([0,\infty),\mathbf{R})$. Hence, the relative compactness of MX (in S) has been established.

Next, we will show that *the mapping M is continuous.* For this purpose, let us consider an arbitrary function x in X and a sequence $\left(x^{[\nu]}\right)_{\nu\ge 1}$ of functions in X with

$$\|\!\cdot\!\| - \lim_{\nu\to\infty} x^{[\nu]} = x.$$

It is not difficult to verify that

$$\lim_{\nu\to\infty} x^{[\nu]}(t)=x(t) \quad \text{uniformly in } t\in[-r,\infty)$$

and

$$\lim_{\nu\to\infty}(x^{[\nu]})'(t)=x'(t) \quad \text{uniformly in } t\in[0,\infty).$$

On the other hand, by (3.8), it holds

$$\left|f(t,x_t^{[\nu]},(x^{[\nu]})'(t))\right| \le F(t,\gamma_t,c) \quad \text{for every } t\ge 0 \quad \text{and} \quad \text{for all } \nu\ge 1.$$

Thus, because of (3.1) and (3.14), one can apply the Lebesgue dominated convergence theorem to obtain, for $t\ge 0$,

$$\lim_{\nu\to\infty}\int_t^\infty (s-t)f(s,x_s^{[\nu]},(x^{[\nu]})'(s))ds=\int_t^\infty (s-t)f(s,x_s,x'(s))ds.$$

So, from (3.3) it follows that

$$\lim_{\nu\to\infty}(Mx^{[\nu]})(t)=(Mx)(t) \quad \text{for } t\ge -r.$$

It remains to establish that this pointwise convergence is also $\nparallel \cdot \nparallel$ −convergence, i.e. that

$$\nparallel \cdot \nparallel - \lim_{\nu\to\infty} Mx^{[\nu]} = Mx. \tag{3.16}$$

To this end, we consider an arbitrary subsequence $\left(Mx^{[\mu_\nu]}\right)_{\nu\geq 1}$ of $\left(Mx^{[\nu]}\right)_{\nu\geq 1}$. Since the set MX is relatively compact, there exist a subsequence $\left(Mx^{[\mu_{\lambda_\nu}]}\right)_{\nu\geq 1}$ of $\left(Mx^{[\mu_\nu]}\right)_{\nu\geq 1}$ and a function u in S so that

$$\nparallel \cdot \nparallel - \lim_{\nu\to\infty} Mx^{[\mu_{\lambda_\nu}]} = u.$$

As the $\nparallel \cdot \nparallel$ −convergence implies the pointwise convergence to the same limit function, we must have $u = Mx$. That is, (3.16) holds true. Consequently, M is continuous.

The proof of the proposition has been completed.

Now, we proceed to the proof of Theorem 1.

Proof of Theorem 1. Let X be defined by (3.2). Clearly, X *is a nonempty convex and closed subset of* S. Assumption (2.2) guarantees, in particular, that (3.1) holds. So, by Proposition 1, the formula (3.3) makes sense for any function x in X, and this formula defines *a continuous mapping* M *of* X *into* S *such that* MX *is relatively compact.* We shall prove that M is a mapping of X *into itself*, i.e. that $MX \subseteq X$. Let us consider an arbitrary function x in X. Then, by taking into account (3.8), from (3.3) we obtain, for $-r \leq t \leq 0$,

$$\begin{aligned} |(Mx)(t) - \eta| &= \left| -\int_0^\infty sf(s, x_s, x'(s))ds \right| \leq \int_0^\infty s\,|f(s, x_s, x'(s))|\, ds \\ &\leq \int_0^\infty sF(s, \gamma_s, c)ds \end{aligned}$$

and consequently, in view of (2.2), we find

$$|(Mx)(t) - \eta| \leq c - |\eta| \quad \text{for } -r \leq t \leq 0. \tag{3.17}$$

Moreover, by using again (3.8), from (3.3) we derive for $t \geq 0$

$$\begin{aligned} |(Mx)'(t) - \xi| &= \left| \int_t^\infty f(s, x_s, x'(s))ds \right| \leq \int_t^\infty |f(s, x_s, x'(s))|\, ds \\ &\leq \int_t^\infty F(s, \gamma_s, c)ds \leq \int_0^\infty F(s, \gamma_s, c)ds \end{aligned}$$

and so, by (2.3), we get

$$|(Mx)'(t) - \xi| \leq c - |\xi| \quad \text{for every } t \geq 0. \tag{3.18}$$

Inequalities (3.17) and (3.18) give

$$|(Mx)(t)| \leq c \ \text{ for } -r \leq t \leq 0$$

and

$$|(Mx)'(t)| \leq c \ \text{ for every } t \geq 0,$$

respectively. From the last two inequalities it follows that $\nparallel Mx \nparallel \leq c$, which means that Mx belongs to X. We have thus proved that $Mx \in X$ for each $x \in X$, i.e. that $MX \subseteq X$.

Now, we apply the Schauder theorem to conclude that there exists at least one x in X with $x = Mx$, i.e.

$$x(t) = \begin{cases} \eta - \int_0^\infty sf(s, x_s, x'(s))ds & \text{for } -r \le t \le 0 \\ \xi t + \eta - \int_t^\infty (s-t) f(s, x_s, x'(s))ds & \text{for } t \ge 0. \end{cases} \tag{3.19}$$

From (3.19) we immediately obtain

$$x''(t) = -f(t, x_t, x'(t)) \quad \text{for all } t \ge 0$$

and so x is a solution on $[0, \infty)$ of the delay differential equation (E). As x belongs to X, (3.11) holds true and consequently

$$\lim_{t\to\infty} \int_t^\infty (s-t) f(s, x_s, x'(s))ds = 0 = \lim_{t\to\infty} \int_t^\infty f(s, x_s, x'(s))ds.$$

By using this fact, from (3.19) we can easily conclude that the solution x is such that (2.5) and (2.6) hold. Furthermore, (2.7) is an immediate consequence of (3.19). Moreover, by taking into account (3.8), from (3.19) we obtain for $t \ge 0$

$$\begin{aligned} |x(t) - (\xi t + \eta)| &= \left| -\int_t^\infty (s-t) f(s, x_s, x'(s))ds \right| \le \int_t^\infty (s-t) \left| f(s, x_s, x'(s)) \right| ds \\ &\le \int_t^\infty (s-t) F(s, \gamma_s, c) ds \le \int_0^\infty sF(s, \gamma_s, c)ds. \end{aligned}$$

Thus, in view of (2.2), we have

$$|x(t) - (\xi t + \eta)| \le c - |\eta| \quad \text{for every } t \ge 0. \tag{3.20}$$

Also, since $x = Mx$, it follows from (3.18) that

$$|x'(t) - \xi| \le c - |\xi| \quad \text{for every } t \ge 0. \tag{3.21}$$

Finally, we see that (3.20) and (3.21) coincide with (2.8) and (2.9), respectively.

The proof of the theorem is complete.

4. PROOF OF THEOREM 2

The proof of Theorem 2 is also based on the use of the Schauder's theorem stated in the previous section. The compactness criterion for subsets of the Banach space $BC([0, \infty), \mathbf{R})$, which is given in Section 3, will also be needed in the present section.

In this section, S_0 stands for the set of all functions u in $C([-r, \infty), \mathbf{R})$ with $u(t) = O(t)$ for $t \to \infty$. The set S_0 *is a Banach space endowed with the norm* $\| \cdot \|_0$ defined by the formula

$$\| u \|_0 = \max \left\{ \max_{-r \le t \le 0} |u(t)|, \ \sup_{t \ge 0} \frac{|u(t)|}{t+1} \right\} \quad \text{for } u \in S_0.$$

The following proposition will be used in order to prove Theorem 2.

Proposition 2. *Assume that* (2.10) *holds, where* F_0 *is a nonnegative real-valued function defined on* $[0, \infty) \times C([-r, 0], [0, \infty))$, *which satisfies the Continuity Condition* (C_0). *Suppose that* (B_0) *is satisfied.*

Let ξ and η be given real constants, and let c be a positive real number such that

$$\int_0^\infty tF_0(t,\gamma_t)dt < \infty, \tag{4.1}$$

where the function γ in $C([-r,\infty),[0,\infty))$ depends on c and is defined by (2.4). Let also X_0 be the subset of S_0 defined by

$$X_0 = \{x \in S_0 : \ \nparallel x \nparallel_0 \leq c\}. \tag{4.2}$$

Then the formula

$$(M_0x)(t) = \begin{cases} \eta - \int_0^\infty sf_0(s,x_s)ds & \textit{for } -r \leq t \leq 0 \\ \xi t + \eta - \int_t^\infty (s-t)f_0(s,x_s)ds & \textit{for } t \geq 0 \end{cases} \tag{4.3}$$

makes sense for any function x in X_0, and this formula defines a continuous mapping M_0 of X_0 into S_0 such that M_0X_0 is relatively compact.

Proof of Proposition 2. Consider an arbitrary function x in X_0. By taking into account the definition, by (4.2), of the set X_0, we immediately see that x satisfies (3.4) and (3.6). These two inequalities together with (2.4) imply $|x(t)| \leq \gamma(t)$ for $t \geq -r$. Consequently, (3.7) holds true. By using (3.7) and the assumption (B_0), we find

$$F_0(t,|x_t|) \leq F_0(t,\gamma_t) \quad \text{for } t \geq 0.$$

But, in view of (2.10), it holds

$$|f_0(t,x_t)| \leq F_0(t,|x_t|) \quad \text{for } t \geq 0.$$

Hence, we have

$$|f_0(t,x_t)| \leq F_0(t,\gamma_t) \quad \text{for every } t \geq 0. \tag{4.4}$$

From (4.1) and (4.4) it follows that

$$\int_0^\infty t\,|f_0(t,x_t)|\,dt < \infty, \tag{4.5}$$

which ensures, in particular, that

$$\int_0^\infty |f_0(t,x_t)|\,dt < \infty. \tag{4.6}$$

Inequalities (4.5) and (4.6) guarantee that

$$\int_0^\infty tf_0(t,x_t)dt \quad \text{and} \quad \int_0^\infty f_0(t,x_t)dt \qquad \text{exist in } \mathbf{R}. \tag{4.7}$$

Since (4.7) holds true for every function x in X_0, we can immediately conclude that *the formula* (4.3) *makes sense for any function x in X_0, and this formula defines a mapping M_0 of X_0* into $C([-r,\infty),\mathbf{R})$. Furthermore, we shall prove that M_0 is a mapping of X_0 *into* S_0, i.e. that $M_0X_0 \subseteq S_0$. For this purpose, let us consider an arbitrary function x in X_0. Then, by taking into account (4.4), from (4.3) we derive for $t \geq 0$

$$\begin{aligned}\frac{|(M_0x)(t)|}{t+1} &= \left|\frac{\xi t+\eta}{t+1} - \frac{1}{t+1}\int_t^\infty (s-t)f_0(s,x_s)ds\right| \\ &\leq \frac{|\xi|\,t+|\eta|}{t+1} + \frac{1}{t+1}\int_t^\infty (s-t)\,|f_0(s,x_s)|\,ds \\ &\leq \max\{|\xi|\,,|\eta|\} + \int_t^\infty (s-t)\,|f_0(s,x_s)|\,ds \\ &\leq \max\{|\xi|\,,|\eta|\} + \int_t^\infty (s-t)F_0(s,\gamma_s)ds \\ &\leq \max\{|\xi|\,,|\eta|\} + \int_0^\infty sF_0(s,\gamma_s)ds.\end{aligned}$$

So, if we set

$$Q_0 = \max\{|\xi|\,,|\eta|\} + \int_0^\infty sF_0(s,\gamma_s)ds, \tag{4.8}$$

then we have

$$\frac{|(M_0x)(t)|}{t+1} \leq Q_0 \quad \text{for all } t \geq 0. \tag{4.9}$$

We note that, because of (4.1), Q_0 is a nonnegative real constant. It follows from (4.9) that M_0x belongs to S_0. Thus, it has been established that $M_0x \in S_0$ for every function $x \in X_0$, i.e. that $M_0X_0 \subseteq S_0$.

Now, we will show that M_0X_0 *is relatively compact.* We observe that, for any function x in X_0, it follows from (4.3) that

$$(M_0x)(t) = (M_0x)(0) = \left.\frac{(M_0x)(s)}{s+1}\right|_{s=0} \quad \text{for } -r \leq t \leq 0.$$

By taking into account this fact as well as the definition of the norm $\|\cdot\|_0$, we can easily see that it is enough to show that the set

$$U_0 = \left\{u: \text{ There exists } x \in X_0 \text{ such that } u(t) = \frac{(M_0x)(t)}{t+1} \text{ for } t \geq 0\right\}$$

is relatively compact in the Banach space $BC([0,\infty),\mathbf{R})$. Every function x in X_0 is such that (4.9) holds, where the nonnegative real constant Q_0 is defined by (4.8) (and it is independent of x). Thus, the set U_0 is uniformly bounded. Furthermore, let x be an arbitrary function in X_0. Then, from (4.3) we obtain for $t \geq 0$

$$\begin{aligned}\left|\frac{(M_0x)(t)}{t+1} - \xi\right| &= \left|\frac{\xi t+\eta - \int_t^\infty (s-t)f_0(s,x_s)ds}{t+1} - \xi\right| \\ &= \frac{\left|-\xi+\eta - \int_t^\infty (s-t)f_0(s,x_s)ds\right|}{t+1} \\ &\leq \frac{|-\xi+\eta| + \int_t^\infty (s-t)\,|f_0(s,x_s)|\,ds}{t+1}.\end{aligned}$$

Hence, in view of (4.4), it holds

$$\left|\frac{(M_0x)(t)}{t+1} - \xi\right| \leq \frac{|-\xi+\eta| + \int_t^\infty (s-t)F_0(s,\gamma_s)ds}{t+1} \quad \text{for } t \geq 0. \tag{4.10}$$

It follows, in particular, from (4.1) that

$$\int_0^\infty F_0(t,\gamma_t)dt < \infty. \tag{4.11}$$

We get

$$\begin{aligned}\lim_{t\to\infty}\frac{|-\xi+\eta|+\int_t^\infty(s-t)F_0(s,\gamma_s)ds}{t+1} &= \lim_{t\to\infty}\left[|-\xi+\eta|+\int_t^\infty(s-t)F_0(s,\gamma_s)ds\right]' \\ &= \lim_{t\to\infty}\left[-\int_t^\infty F_0(s,\gamma_s)ds\right]\end{aligned}$$

and, consequently, by virtue of (4.11), we find

$$\lim_{t\to\infty}\frac{|-\xi+\eta|+\int_t^\infty(s-t)F_0(s,\gamma_s)ds}{t+1}=0. \tag{4.12}$$

Inequality (4.10) together with (4.12) implies

$$\lim_{t\to\infty}\frac{(M_0x)(t)}{t+1}=\xi.$$

By using again (4.10) and (4.12), we can easily conclude that U_0 is equiconvergent at ∞. Now, let again x be an arbitrary function in X_0. From (4.3) we derive for every $t\geq 0$

$$\begin{aligned}\left|\left[\frac{(M_0x)(t)}{t+1}\right]'\right| &= \frac{1}{(t+1)^2}\left|(t+1)(M_0x)'(t)-(M_0x)(t)\right| \\ &\leq \left|(t+1)(M_0x)'(t)-(M_0x)(t)\right| \\ &= \Bigg|(t+1)\left[\xi+\int_t^\infty f_0(s,x_s)ds\right] \\ &\quad -\left[\xi t+\eta-\int_t^\infty(s-t)f_0(s,x_s)ds\right]\Bigg| \\ &= \left|\xi-\eta+\int_t^\infty f_0(s,x_s)ds+\int_t^\infty sf_0(s,x_s)ds\right| \\ &\leq |\xi-\eta|+\int_t^\infty|f_0(s,x_s)|\,ds+\int_t^\infty s\,|f_0(s,x_s)|\,ds \\ &\leq |\xi-\eta|+\int_0^\infty|f_0(s,x_s)|\,ds+\int_0^\infty s\,|f_0(s,x_s)|\,ds.\end{aligned}$$

So, because of (4.4), we have

$$\left|\left[\frac{(M_0x)(t)}{t+1}\right]'\right|\leq\Theta\quad\text{for every } t\geq 0, \tag{4.13}$$

where

$$\Theta=|\xi-\eta|+\int_0^\infty F_0(s,\gamma_s)ds+\int_0^\infty sF_0(s,\gamma_s)ds.$$

In view of (4.1) and (4.11), Θ is a nonnegative real number. By taking into account (4.13) and applying the mean value theorem, we find

$$\left|\frac{(M_0x)(t_1)}{t_1+1}-\frac{(M_0x)(t_2)}{t_2+1}\right|\leq\Theta\,|t_1-t_2|\quad\text{for every } t_1\geq 0,\ t_2\geq 0.$$

Since the last inequality is fulfilled for all functions x in X_0 (and Θ is independent of x), we immediately see that U_0 is equicontinuous. By the given compactness criterion, U_0 is relatively compact in $BC\left([0,\infty),\mathbf{R}\right)$. So, the relative compactness of M_0X_0 has been proved.

Next, we shall prove that *the mapping* M_0 *is continuous.* Let x be an arbitrary function in X_0 and $\left(x^{[\nu]}\right)_{\nu\geq 1}$ be any sequence of functions in X_0 with

$$\nparallel \cdot \nparallel_0 - \lim_{\nu\to\infty} x^{[\nu]} = x.$$

It is not difficult to verify that

$$\|\cdot\| - \lim_{\nu\to\infty} x_t^{[\nu]} = x_t \quad \text{for every } t \geq 0.$$

Moreover, (4.4) guarantees that

$$\left|f_0(t, x_t^{[\nu]})\right| \leq F_0(t,\gamma_t) \quad \text{for every } t\geq 0 \quad \text{and} \quad \text{for all } \nu \geq 1.$$

So, by taking into account (4.1) and (4.11), we can apply the Lebesgue dominated convergence theorem to obtain, for $t \geq 0$,

$$\lim_{\nu\to\infty}\int_t^\infty (s-t)f_0(s,x_s^{[\nu]})ds = \int_t^\infty (s-t)f_0(s,x_s)ds.$$

Thus, from (4.3) it follows that

$$\lim_{\nu\to\infty}(M_0x^{[\nu]})(t) = (M_0x)(t) \quad \text{for } t\geq -r.$$

Since M_0X_0 is relatively compact and the $\nparallel \cdot \nparallel_0$ −convergence implies the pointwise convergence to the same limit function, we can follow the same procedure as in the proof of Proposition 1 to conclude that the above convergence is also $\nparallel \cdot \nparallel_0$ −convergence, i.e. to conclude that

$$\nparallel \cdot \nparallel_0 - \lim_{\nu\to\infty} M_0x^{[\nu]} = M_0x.$$

This shows that M_0 is continuous.

The proof of the proposition is now complete.

Now, we proceed to the proof of Theorem 2.

Proof of Theorem 2. Consider the set X_0 defined by (4.2). It is clear that X_0 *is a nonempty convex and closed subset of* S_0. It follows, in particular, from the hypothesis (2.11) that (4.1) holds. Hence, Proposition 2 guarantees that the formula (4.3) makes sense for any function x in X_0, and this formula defines *a continuous mapping* M_0 *of* X_0 *into* S_0 *such that* M_0X_0 *is relatively compact.* We will show that M_0 is a mapping of X_0 *into itself*, i.e. that $M_0X_0 \subseteq X_0$. For this purpose, let us consider an arbitrary function x in X_0. Then (4.9) is satisfied, where the nonnegative real number Q_0 is defined by (4.8). Assumption (2.11) ensures that $Q_0 \leq c$. So, (4.9) gives

$$\frac{|(M_0x)(t)|}{t+1} \leq c \quad \text{for every } t\geq 0. \tag{4.14}$$

In particular, (4.14) guarantees that $|(M_0x)(0)| \leq c$. But, from (4.3) it follows that M_0x is constant on the interval $[-r,0]$. So, we always have

$$|(M_0x)(t)| \leq c \quad \text{for } -r \leq t \leq 0. \tag{4.15}$$

Inequalities (4.14) and (4.15) give $\nparallel M_0x \nparallel_0 \leq c$, which means that M_0x belongs to X_0. So, we have proved that $M_0x \in X_0$ for every function x in X_0, which ensures that $M_0X_0 \subseteq X_0$.

Now, by applying the Schauder theorem, we conclude that there exists at least one x in X_0 with $x = M_0x$, i.e.

$$(4.16) \qquad x(t) = \begin{cases} \eta - \int_0^\infty sf_0(s,x_s)ds & \text{for } -r \leq t \leq 0 \\ \xi t + \eta - \int_t^\infty (s-t)f_0(s,x_s)ds & \text{for } t \geq 0. \end{cases}$$

We immediately obtain

$$x''(t) = -f_0(t,x_t) \quad \text{for } t \geq 0$$

and consequently x is a solution on $[0,\infty)$ of the delay differential equation (E_0). Since $x \in X_0$, (4.7) is true and so

$$\lim_{t\to\infty}\int_t^\infty (s-t)f_0(s,x_s)ds = 0 = \lim_{t\to\infty}\int_t^\infty f_0(s,x_s)ds.$$

By taking into account this fact, we can use (4.16) to see that the solution x is such that (2.5) and (2.6) hold. Next, we observe that (2.7) is an immediate consequence of (4.16). Furthermore, by taking into account (4.4), from (4.16) we get for $t \geq 0$

$$\begin{aligned} |x(t) - (\xi t + \eta)| &= \left| -\int_t^\infty (s-t)f_0(s,x_s)ds \right| \leq \int_t^\infty (s-t)\,|f_0(s,x_s)|\,ds \\ &\leq \int_t^\infty (s-t)F_0(s,\gamma_s)ds \leq \int_0^\infty sF_0(s,\gamma_s)ds. \end{aligned}$$

So, because of (2.11), it holds

$$(4.17) \qquad |x(t) - (\xi t + \eta)| \leq c - \max\{|\xi|, |\eta|\} \quad \text{for every } t \geq 0.$$

Moreover, (4.16) gives, for $t \geq 0$,

$$|x'(t) - \xi| = \left|\int_t^\infty f_0(s,x_s)ds\right| \leq \int_t^\infty |f_0(s,x_s)|\,ds \leq \int_0^\infty |f_0(s,x_s)|\,ds.$$

Therefore, by (4.4), we have

$$(4.18) \qquad |x'(t) - \xi| \leq \int_0^\infty F_0(s,\gamma_s)ds \quad \text{for every } t \geq 0.$$

Note that, because of (4.11), $\int_0^\infty F_0(s,\gamma_s)ds$ is finite. Finally, we see that (4.17) and (4.18) coincide with (2.12) and (2.13), respectively.

The proof of the theorem is complete.

5. PROOFS OF THEOREMS 3 AND 4

In order to prove Theorems 3 and 4, we will make use of the well-known Banach's Contraction Principle (see, e.g., Kartsatos [7]).

The Banach Contraction Principle. *Let P be a Banach space and Y any nonempty closed subset of P. If N is a contraction of Y into itself, then the mapping N has exactly one fixed point (i.e. there exists a unique $y \in Y$ with $y = Ny$).*

The following lemma provides a useful integral representation of Problem (D), (2.5), (2.6) (where ξ and η are given real constants), which will be used in proving Theorem 3.

Lemma 1. *Let ξ and η be given real constants. A real-valued function x, which is continuously differentiable on the interval $[0,\infty)$, is a solution on $[0,\infty)$ of the ordinary differential equation* (D) *such that* (2.5) *and* (2.6) *hold, if and only if it satisfies*

$$x(t) = \xi t + \eta - \int_t^\infty (s-t)h(s,x(s),x'(s))ds \quad \text{for } t \geq 0. \tag{5.1}$$

A particular case of Lemma 1 is Lemma 2 below concerning Problem (D_0), (2.5), (2.6); Lemma 2 will be used in the proof of Theorem 4.

Lemma 2. *Let ξ and η be given real constants. A function x in $C([0,\infty),\mathbf{R})$ is a solution on $[0,\infty)$ of the ordinary differential equation* (D_0) *such that* (2.5) *and* (2.6) *hold, if and only if it satisfies*

$$x(t) = \xi t + \eta - \int_t^\infty (s-t)h_0(s,x(s))ds \quad \text{for } t \geq 0. \tag{5.2}$$

Proof of Lemma 1. Let x be a real-valued function, which is continuously differentiable on the interval $[0,\infty)$.

Assume first that x satisfies (5.1). Then

$$\lim_{t\to\infty}[x(t) - (\xi t + \eta)] = -\lim_{t\to\infty}\int_t^\infty (s-t)h(s,x(s),x'(s))ds = 0$$

and so (2.5) holds true. Also, we immediately obtain

$$x'(t) = \xi + \int_t^\infty h(s,x(s),x'(s))ds \quad \text{for every } t \geq 0,$$

which gives

$$\lim_{t\to\infty}[x'(t) - \xi] = \lim_{t\to\infty}\int_t^\infty h(s,x(s),x'(s))ds = 0,$$

i.e. (2.6) is fulfilled. Moreover, we have

$$x''(t) = -h(t,x(t),x'(t)) \quad \text{for all } t \geq 0,$$

which means that x is a solution on $[0,\infty)$ of (D).

Conversely, let us suppose that x is a solution on $[0,\infty)$ of (D) such that (2.5) and (2.6) hold. Then from (D) it follows that

$$x'(T) - x'(t) = -\int_t^T h(s,x(s),x'(s))ds \quad \text{for all } T,\ t \text{ with } T \geq t \geq 0.$$

Consequently,

$$\lim_{T\to\infty} x'(T) - x'(t) = -\int_t^\infty h(s,x(s),x'(s))ds \quad \text{for every } t \geq 0.$$

But, in view of (2.6), we have $\lim_{T\to\infty} x'(T) = \xi$. Thus,

$$x'(t) = \xi + \int_t^\infty h(s, x(s), x'(s))ds \quad \text{for } t \geq 0.$$

This gives

$$x(T) - x(t) = \xi(T-t) + \int_t^T \int_s^\infty h(\sigma, x(\sigma), x'(\sigma))d\sigma ds \quad \text{for } T \geq t \geq 0,$$

which can equivalently be written as

$$[x(T) - (\xi T + \eta)] - [x(t) - (\xi t + \eta)] = \int_t^T \int_s^\infty h(\sigma, x(\sigma), x'(\sigma))d\sigma ds \quad \text{for } T \geq t \geq 0.$$

Hence,

$$\lim_{T\to\infty} [x(T) - (\xi T + \eta)] - [x(t) - (\xi t + \eta)] = \int_t^\infty \int_s^\infty h(\sigma, x(\sigma), x'(\sigma))d\sigma ds$$
$$= \int_t^\infty (s-t)h(s, x(s), x'(s))ds \quad \text{for } t \geq 0.$$

But, because of (2.5), it holds $\lim_{T\to\infty} [x(T) - (\xi T + \eta)] = 0$. Therefore,

$$-x(t) + (\xi t + \eta) = \int_t^\infty (s-t)h(s, x(s), x'(s))ds \quad \text{for all } t \geq 0,$$

i.e. x satisfies (5.1).

The proof of the lemma has been finished.

Now, we are in a position to present the proofs of Theorems 3 and 4.

Proof of Theorem 3. Let P be the set of all real-valued functions on the interval $[0, \infty)$, which have bounded continuous derivatives on $[0, \infty)$. This set *is a Banach space endowed with the norm* $\|\cdot\|^*$ defined by

$$\|u\|^* = \max\left\{|u(0)|, \sup_{t\geq 0}|u'(t)|\right\} \quad \text{for } u \in P.$$

Let also Y be the *nonempty closed subset of* P defined by

$$Y = \{x \in P : \ \|x\|^* \leq c\}.$$

Clearly, Y is the subset of P consisting of all functions x in P which satisfy (2.24) and (2.25).

Consider now an arbitrary function x in Y. Then x satisfies (2.24) and (2.25), which imply (2.28). By using (2.25) and (2.28) as well as the hypothesis (A), we obtain

$$H(t, |x(t)|, |x'(t)|) \leq H(t, c(t+1), c) \quad \text{for } t \geq 0.$$

On the other hand, the assumption (2.16) guarantees that

$$|h(t, x(t), x'(t))| \leq H(t, |x(t)|, |x'(t)|) \quad \text{for } t \geq 0.$$

Thus, we have

$$|h(t, x(t), x'(t))| \leq H(t, c(t+1), c) \quad \text{for all } t \geq 0. \tag{5.3}$$

Furthermore, we observe that the hypothesis (2.17) ensures, in particular, that

$$\int_0^\infty tH(t, c(t+1), c)dt < \infty$$

and consequently, by taking into account (5.3), we obtain

$$\int_0^\infty t\,|h(t, x(t), x'(t))|\,dt < \infty.$$

So,

$$\int_0^\infty th(t, x(t), x'(t))dt \quad \text{and, in particular,} \quad \int_0^\infty h(t, x(t), x'(t))dt \qquad \text{exist in } \mathbf{R}.$$

This is true for all functions x in Y. Hence, the formula

$$(Nx)(t) = \xi t + \eta - \int_t^\infty (s-t)h(s, x(s), x'(s))ds \quad \text{for } t \geq 0$$

makes sense for any function x in Y, and this formula defines *a mapping N of Y* into $C([0,\infty),\mathbf{R})$. We will show that N is a mapping of Y *into itself*, i.e. that $NY \subseteq Y$. To this end, let us consider an arbitrary function x in Y. Then, by taking into account (5.3), we obtain

$$\begin{aligned} |(Nx)(0)| &= \left|\eta - \int_0^\infty sh(s, x(s), x'(s))ds\right| \leq |\eta| + \int_0^\infty s\,|h(s, x(s), x'(s))|\,ds \\ &\leq |\eta| + \int_0^\infty sH(s, c(s+1), c)ds \end{aligned}$$

and consequently, in view of (2.17), it holds

$$|(Nx)(0)| \leq c. \tag{5.4}$$

Furthermore, by taking again into account (5.3), we derive for $t \geq 0$

$$\begin{aligned} |(Nx)'(t) - \xi| &= \left|\int_t^\infty h(s, x(s), x'(s))ds\right| \leq \int_t^\infty |h(s, x(s), x'(s))|\,ds \\ &\leq \int_t^\infty H(s, c(s+1), c)ds \leq \int_0^\infty H(s, c(s+1), c)ds \end{aligned}$$

and so, because of (2.18), we have

$$|(Nx)'(t) - \xi| \leq c - |\xi| \quad \text{for all } t \geq 0. \tag{5.5}$$

It follows from (5.5) that

$$|(Nx)'(t)| \leq c \ \text{ for every } t \geq 0. \tag{5.6}$$

Inequalities (5.4) and (5.6) mean that Nx belongs to Y. It has been verified that, for each $x \in Y$, Nx belongs to Y. Thus, we always have $NY \subseteq Y$.

Now, let u be an arbitrary function in P. Then

$$|u(0)| \leq \|u\|^* \tag{5.7}$$

and

$$|u'(t)| \leq \|u\|^* \quad \text{for every } t \geq 0. \tag{5.8}$$

Furthermore, by using (5.7) and (5.8), we can immediately see that u is also such that

$$|u(t)| \leq \|u\|^*\,(t+1) \quad \text{for every } t \geq 0. \tag{5.9}$$

Next, let us consider two arbitrary functions x and y in Y. Then, by using the assumption (2.22) and taking into account (5.9) and (5.8), we obtain

$$\begin{aligned}
|(Nx)(0)-(Ny)(0)| &= \left|-\int_0^\infty s\left[h(s,x(s),x'(s))-h(s,y(s),y'(s))\right]ds\right| \\
&\le \int_0^\infty s\left|h(s,x(s),x'(s))-h(s,y(s),y'(s))\right|ds \\
&\le \int_0^\infty sL(s)\max\left\{|x(s)-y(s)|\,,|x'(s)-y'(s)|\right\}ds \\
&\le \int_0^\infty sL(s)\max\left\{\|x-y\|^*(s+1),\|x-y\|^*\right\}ds \\
&= \left[\int_0^\infty sL(s)\max\{s+1,1\}\,ds\right]\|x-y\|^*.
\end{aligned}$$

That is,

$$|(Nx)(0)-(Ny)(0)| \le \left[\int_0^\infty s(s+1)L(s)ds\right]\|x-y\|^*. \tag{5.10}$$

Furthermore, by using again (2.22) and taking again into account (5.9) and (5.8), we get for $t \ge 0$

$$\begin{aligned}
|(Nx)'(t)-(Ny)'(t)| &= \left|\int_t^\infty \left[h(s,x(s),x'(s))-h(s,y(s),y'(s))\right]ds\right| \\
&\le \int_t^\infty \left|h(s,x(s),x'(s))-h(s,y(s),y'(s))\right|ds \\
&\le \int_0^\infty \left|h(s,x(s),x'(s))-h(s,y(s),y'(s))\right|ds \\
&\le \int_0^\infty L(s)\max\left\{|x(s)-y(s)|\,,|x'(s)-y'(s)|\right\}ds \\
&\le \int_0^\infty L(s)\max\left\{\|x-y\|^*(s+1),\|x-y\|^*\right\}ds \\
&= \left[\int_0^\infty L(s)\max\{s+1,1\}\,ds\right]\|x-y\|^* \\
&= \left[\int_0^\infty (s+1)L(s)ds\right]\|x-y\|^*.
\end{aligned}$$

Thus, we find

$$\sup_{t\ge 0}|(Nx)'(t)-(Ny)'(t)| \le \left[\int_0^\infty (s+1)L(s)ds\right]\|x-y\|^*. \tag{5.11}$$

Set

$$\theta = \max\left\{\int_0^\infty s(s+1)L(s)ds,\ \int_0^\infty (s+1)L(s)ds\right\}.$$

Then (5.10) and (5.11) give

$$\|Nx-Ny\|^* \le \theta\,\|x-y\|^*.$$

This inequality holds true for all functions x and y in Y. On the other hand, from the hypothesis (2.23) it follows that $0 \le \theta < 1$. We have thus proved that *the mapping $M : Y \longrightarrow Y$ is a contraction.*

Finally, by using the Banach Contraction Principle, we conclude that there exists exactly one function x in Y with $x = Nx$. We see that $x = Nx$ is equivalent to the fact that x satisfies (5.1). Hence, by Lemma 1, the ordinary differential equation (D) has exactly one solution x on the interval $[0, \infty)$ satisfying (2.24) and (2.25), and such that (2.5) and (2.6) hold. It remains to establish that this unique solution x satisfies (2.8) and (2.9). By taking into account (5.3), from (5.1) we obtain for $t \geq 0$

$$\begin{aligned} & |x(t) - (\xi t + \eta)| \\ = & \left| -\int_t^\infty (s-t)h(s, x(s), x'(s))ds \right| \leq \int_t^\infty (s-t)\,|h(s, x(s), x'(s))|\,ds \\ \leq & \int_t^\infty (s-t)H(s, c(s+1), c)ds \leq \int_0^\infty sH(s, c(s+1), c)ds. \end{aligned}$$

So, by using (2.17), we immediately arrive at (3.20). Moreover, as $x = Nx$, it follows from (5.5) that x satisfies (3.21). We see that (3.20) and (3.21) coincide with (2.8) and (2.9), respectively.

The proof of the theorem is complete.

Proof of Theorem 4. Consider the set P_0 of all continuous real-valued functions u on the interval $[0, \infty)$ with $u(t) = O(t)$ for $t \to \infty$. The set P_0 *is a Banach space endowed with the norm* $\|\cdot\|_0^*$ defined by

$$\|u\|_0^* = \sup_{t \geq 0} \frac{|u(t)|}{t+1} \quad \text{for } u \in P_0.$$

Consider also the set Y_0 defined by

$$Y_0 = \left\{ x \in P_0 : \; \|x\|_0^* \leq c \right\}.$$

It is clear that Y_0 is the subset of P_0 consisting of all functions x in P_0 which satisfy (2.28). The set Y_0 *is a nonempty closed subset of* P_0.

Now, let x be an arbitrary function in Y_0. Then x satisfies (2.28). By taking into account (2.28) and using the assumption (A_0), we get

$$H_0(t, |x(t)|) \leq H_0(t, c(t+1)) \quad \text{for } t \geq 0.$$

But, from the hypothesis (2.19) it follows that

$$|h_0(t, x(t))| \leq H_0(t, |x(t)|) \quad \text{for } t \geq 0.$$

By combining the last two inequalities, we obtain

$$|h_0(t, x(t))| \leq H_0(t, c(t+1)) \quad \text{for all } t \geq 0. \tag{5.12}$$

Next, we see that (2.20) implies, in particular,

$$\int_0^\infty tH_0(t, c(t+1))dt < \infty.$$

Thus, because of (5.12), we have

$$\int_0^\infty t\,|h_0(t, x(t))|\,dt < \infty,$$

which guarantees that

$$\int_0^\infty th_0(t, x(t))dt \quad \text{and, in particular,} \quad \int_0^\infty h_0(t, x(t))dt \qquad \text{exist in } \mathbf{R}.$$

So, as the function x in Y_0 is arbitrary, we immediately see that the formula

$$(N_0x)(t) = \xi t + \eta - \int_t^\infty (s-t)h_0(s,x(s))ds \quad \text{for } t \geq 0$$

makes sense for any function x in Y_0, and this formula defines *a mapping* N_0 *of* Y_0 into $C([0,\infty),\mathbf{R})$. Furthermore, N_0 is a mapping of Y_0 *into itself*, i.e. it holds $N_0Y_0 \subseteq Y_0$. Indeed, by using (5.12) and the hypothesis (2.20), for any function x in Y_0, we obtain, for every $t \geq 0$,

$$\begin{aligned}
\frac{|(N_0x)(t)|}{t+1} &= \left|\frac{\xi t+\eta}{t+1} - \frac{1}{t+1}\int_t^\infty (s-t)h_0(s,x(s))ds\right| \\
&\leq \frac{|\xi|\, t + |\eta|}{t+1} + \frac{1}{t+1}\int_t^\infty (s-t)\,|h_0(s,x(s))|\, ds \\
&\leq \max\{|\xi|\,,|\eta|\} + \int_t^\infty (s-t)\,|h_0(s,x(s))|\, ds \\
&\leq \max\{|\xi|\,,|\eta|\} + \int_t^\infty (s-t)H_0(s,c(s+1))ds \\
&\leq \max\{|\xi|\,,|\eta|\} + \int_0^\infty sH_0(s,c(s+1))ds \\
&\leq c.
\end{aligned}$$

That is, for any $x \in Y_0$, N_0x belongs to Y_0. This proves our assertion.

Furthermore, let x and y be two arbitrary functions in Y_0. Then, by using the hypothesis (2.26), we get for $t \geq 0$

$$\begin{aligned}
\frac{|(N_0x)(t) - (N_0y)(t)|}{t+1} &= \frac{1}{t+1}\left|-\int_t^\infty (s-t)\left[h_0(s,x(s)) - h_0(s,y(s))\right] ds\right| \\
&\leq \frac{1}{t+1}\int_t^\infty (s-t)\,|h_0(s,x(s)) - h_0(s,y(s))|\, ds \\
&\leq \frac{1}{t+1}\int_t^\infty (s-t)L_0(s)\,|x(s)-y(s)|\, ds \\
&= \frac{1}{t+1}\int_t^\infty (s-t)(s+1)L_0(s)\frac{|x(s)-y(s)|}{s+1}ds \\
&\leq \left[\frac{1}{t+1}\int_t^\infty (s-t)(s+1)L_0(s)ds\right]\sup_{s\geq 0}\frac{|x(s)-y(s)|}{s+1}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sup_{t\geq 0}\frac{|(N_0x)(t) - (N_0y)(t)|}{t+1} &\leq \left[\sup_{t\geq 0}\frac{1}{t+1}\int_t^\infty (s-t)(s+1)L_0(s)ds\right]\sup_{s\geq 0}\frac{|x(s)-y(s)|}{s+1} \\
&= \left[\int_0^\infty s(s+1)L_0(s)ds\right]\sup_{s\geq 0}\frac{|x(s)-y(s)|}{s+1}.
\end{aligned}$$

That is,

$$\|N_0x - N_0y\|_0^* \leq \theta_0\,\|x-y\|_0^*\,, \tag{5.13}$$

where

$$\theta_0 = \int_0^\infty s(s+1)L_0(s)ds.$$

Because of the assumption (2.27), we have $0 \leq \theta_0 < 1$. As (5.13) holds true for all functions x and y in Y_0, we conclude that *the mapping* $N_0 : Y_0 \longrightarrow Y_0$ *is a contraction.*

By the Banach Contraction Principle, there exists exactly one function x in Y_0 with $x = N_0x$. Clearly, $x = N_0x$ is equivalent to (5.2). So, from Lemma 2 it follows that the ordinary differential equation (D_0) has exactly one solution x on the interval $[0, \infty)$ satisfying (2.28), and such that (2.5) and (2.6) hold. Finally, we will show that this unique solution x satisfies also (2.12) and (2.21). By taking into account (5.12), from (5.2) we obtain for $t \geq 0$

$$\begin{aligned} |x(t) - (\xi t + \eta)| &= \left| -\int_t^\infty (s-t)h_0(s, x(s))ds \right| \leq \int_t^\infty (s-t)\left|h_0(s, x(s))\right| ds \\ &\leq \int_t^\infty (s-t)H_0(s, c(s+1))ds \leq \int_0^\infty sH_0(s, c(s+1))ds. \end{aligned}$$

Thus, by using (2.20), we arrive at (4.17). Furthermore, it follows from (5.2) that, for $t \geq 0$,

$$|x'(t) - \xi| = \left| \int_t^\infty h_0(s, x(s))ds \right| \leq \int_t^\infty |h_0(s, x(s))| \, ds \leq \int_0^\infty |h_0(s, x(s))| \, ds$$

and consequently, in view of (5.12), we obtain

$$|x'(t) - \xi| \leq \int_0^\infty H_0(s, c(s+1))ds \quad \text{for every } t \geq 0. \tag{5.14}$$

We notice that, because of (2.20), $\int_0^\infty H_0(s, c(s+1))ds$ is finite. We immediately observe that (4.17) and (5.14) coincide with (2.12) and (2.21), respectively.

So, the proof of the theorem has been completed.

6. APPLICATION TO DIFFERENTIAL EQUATIONS OF EMDEN-FOWLER TYPE. EXAMPLES

Consider the second order nonlinear delay differential equations of Emden-Fowler type

$$x''(t) + a(t)\left|x(t-r)\right|^\alpha \operatorname{sgn}x(t-r) + b(t)\left|x'(t)\right|^\beta \operatorname{sgn}x'(t) = 0 \tag{6.1}$$

and

$$x''(t) + a(t)\left|x(t-r)\right|^\alpha \operatorname{sgn}x(t-r) = 0 \tag{6.2}$$

as well as the second order nonlinear ordinary Emden-Fowler differential equations

$$x''(t) + a(t)\left|x(t)\right|^\alpha \operatorname{sgn}x(t) + b(t)\left|x'(t)\right|^\beta \operatorname{sgn}x'(t) = 0 \tag{6.3}$$

and

$$x''(t) + a(t)\left|x(t)\right|^\alpha \operatorname{sgn}x(t) = 0, \tag{6.4}$$

where *a and b are continuous real-valued functions on the interval* $[0, \infty)$, *and α and β are positive real numbers.* Consider also the second order linear ordinary differential equations

$$x''(t) + a(t)x(t) + b(t)x'(t) = 0 \tag{6.5}$$

and

$$x''(t) + a(t)x(t) = 0. \tag{6.6}$$

By applying Theorem 1 (or, especially, Corollary 1) and Theorem 2 (or, especially, Corollary 2) to the delay differential equations (6.1) and (6.2), respectivelly, we are led to the following two results:

Result 1. *Let ξ and η be given real constants, and let there exist a real number c with $c > \max\{|\xi|, |\eta|\}$ so that*

$$c^\alpha \left[\int_0^r t\,|a(t)|\,dt + \int_r^\infty t(t-r+1)^\alpha\,|a(t)|\,dt \right] + c^\beta \int_0^\infty t\,|b(t)|\,dt \le c - |\eta| \tag{6.7}$$

and

$$c^\alpha \left[\int_0^r |a(t)|\,dt + \int_r^\infty (t-r+1)^\alpha\,|a(t)|\,dt \right] + c^\beta \int_0^\infty |b(t)|\,dt \le c - |\xi|. \tag{6.8}$$

Then the delay differential equation (6.1) *has at least one solution x on the interval* $[0,\infty)$ *such that* (2.5) *and* (2.6) *hold; in addition, this solution x satisfies* (2.7), (2.8), *and* (2.9).

Result 2. *Let ξ and η be given real constants, and let there exist a real number c with $c > \max\{|\xi|, |\eta|\}$ so that*

$$c^\alpha \left[\int_0^r t\,|a(t)|\,dt + \int_r^\infty t(t-r+1)^\alpha\,|a(t)|\,dt \right] \le c - \max\{|\xi|, |\eta|\}. \tag{6.9}$$

Then the delay differential equation (6.2) *has at least one solution x on the interval* $[0,\infty)$ *such that* (2.5) *and* (2.6) *hold; in addition, this solution x satisfies* (2.7), (2.12), *and*

$$\xi - c^\alpha \left[\int_0^r |a(s)|\,ds + \int_r^\infty (s-r+1)^\alpha\,|a(s)|\,ds \right] \le x'(t)$$
$$\le \xi + c^\alpha \left[\int_0^r |a(s)|\,ds + \int_r^\infty (s-r+1)^\alpha\,|a(s)|\,ds \right] \quad \text{for every } t \ge 0.$$

(*Note that, because of* (6.9), $\int_r^\infty (s-r+1)^\alpha\,|a(s)|\,ds$ *is finite.*)

Also, an application of Theorem 1 (or, especially, of Corollary 3) and of Theorem 2 (or, especially, of Corollary 4) to the ordinary differential equations (6.3) and (6.4), respectivelly, leads to the next two results:

Result 3. *Let ξ and η be given real constants, and let there exist a real number c with $c > \max\{|\xi|, |\eta|\}$ so that*

$$c^\alpha \int_0^\infty t(t+1)^\alpha\,|a(t)|\,dt + c^\beta \int_0^\infty t\,|b(t)|\,dt \le c - |\eta| \tag{6.10}$$

and

$$c^\alpha \int_0^\infty (t+1)^\alpha\,|a(t)|\,dt + c^\beta \int_0^\infty |b(t)|\,dt \le c - |\xi|. \tag{6.11}$$

Then the ordinary differential equation (6.3) *has at least one solution x on the interval* $[0,\infty)$ *such that* (2.5) *and* (2.6) *hold; in addition, this solution x satisfies* (2.8) *and* (2.9).

Result 4. *Let ξ and η be given real constants, and let there exist a real number c with $c > \max\{|\xi|, |\eta|\}$ so that*

$$c^{\alpha} \int_{0}^{\infty} t(t+1)^{\alpha} |a(t)| \, dt \leq c - \max\{|\xi|, |\eta|\}. \tag{6.12}$$

Then the ordinary differential equation (6.4) has at least one solution x on the interval $[0, \infty)$ such that (2.5) and (2.6) hold; in addition, this solution x satisfies (2.12) and

$$\xi - c^{\alpha} \int_{0}^{\infty} (s+1)^{\alpha} |a(s)| \, ds \leq x'(t) \leq \xi + c^{\alpha} \int_{0}^{\infty} (s+1)^{\alpha} |a(s)| \, ds \quad \text{for every } t \geq 0.$$

(Note that, because of (6.12), $\int_0^\infty (s+1)^\alpha |a(s)| \, ds$ is finite.)

Moreover, if we apply Theorem 3 and Theorem 4 to the linear ordinary differential equations (6.5) and (6.6), respectivelly, then we can arrive at the following two results:

Result 5. *Assume that*

$$\max\left\{ \int_{0}^{\infty} t(t+1) \left[|a(t)| + |b(t)|\right] dt, \ \int_{0}^{\infty} (t+1) \left[|a(t)| + |b(t)|\right] dt \right\} < 1. \tag{6.13}$$

Let ξ and η be given real constants, and let there exist a real number c with $c > \max\{|\xi|, |\eta|\}$ so that

$$c \left[\int_{0}^{\infty} t(t+1) |a(t)| \, dt + \int_{0}^{\infty} t \, |b(t)| \, dt \right] \leq c - |\eta| \tag{6.14}$$

and

$$c \left[\int_{0}^{\infty} (t+1) |a(t)| \, dt + \int_{0}^{\infty} |b(t)| \, dt \right] \leq c - |\xi| . \tag{6.15}$$

Then the linear ordinary differential equation (6.5) has exactly one solution x on the interval $[0, \infty)$ satisfying (2.24) and (2.25), and such that (2.5) and (2.6) hold; in addition, this unique solution x satisfies (2.8) and (2.9).

Result 6. *Assume that*

$$\int_{0}^{\infty} t(t+1) |a(t)| \, dt < 1. \tag{6.16}$$

Let ξ and η be given real constants, and let there exist a real number c with $c > \max\{|\xi|, |\eta|\}$ so that

$$c \int_{0}^{\infty} t(t+1) |a(t)| \, dt \leq c - \max\{|\xi|, |\eta|\}. \tag{6.17}$$

Then the linear ordinary differential equation (6.6) has exactly one solution x on the interval $[0, \infty)$ satisfying (2.28), and such that (2.5) and (2.6) hold; in addition, this unique solution x satisfies (2.12) and

$$\xi - c \int_{0}^{\infty} (s+1) |a(s)| \, ds \leq x'(t) \leq \xi + c \int_{0}^{\infty} (s+1) |a(s)| \, ds \quad \text{for every } t \geq 0.$$

(Note that, because of (6.17), $\int_0^\infty (s+1) |a(s)| \, ds$ is finite.)

Note: Provided that at least one of ξ and η is nonzero, the assumption (6.17) implies the hypothesis (6.16).

Now, we will give some examples to demonstrate the applicability of our results.

Example 1. Consider the delay differential equation (6.1) with $r = 1$, $\alpha = 2$, $\beta = 1$, and

$$a(t) = \frac{8}{21(t+1)^5} \quad \text{for } t \geq 0, \qquad b(t) = \frac{1}{3(t+1)^3} \quad \text{for } t \geq 0.$$

Take $\xi = \frac{5}{6}$ and $\eta = 1$. Inequality (6.7) becomes

$$c^2 \left[\int_0^1 t \frac{8}{21(t+1)^5} dt + \int_1^\infty t^3 \frac{8}{21(t+1)^5} dt \right] + c \int_0^\infty t \frac{1}{3(t+1)^3} dt \leq c - 1,$$

i.e.

$$\frac{1}{9}c^2 + \frac{1}{6}c \leq c - 1. \tag{6.18}$$

We immediately see that (6.18) holds if and only if

$$\frac{3}{2} \leq c \leq 6. \tag{6.19}$$

Furthermore, Inequality (6.8) is written as

$$c^2 \left[\int_0^1 \frac{8}{21(t+1)^5} dt + \int_1^\infty t^2 \frac{8}{21(t+1)^5} dt \right] + c \int_0^\infty \frac{1}{3(t+1)^3} dt \leq c - \frac{5}{6},$$

i.e.

$$\frac{1}{9}c^2 + \frac{1}{6}c \leq c - \frac{5}{6}. \tag{6.20}$$

We observe that (6.20) is satisfied if and only if

$$\frac{15 - \sqrt{105}}{4} \leq c \leq \frac{15 + \sqrt{105}}{4}. \tag{6.21}$$

Since

$$1 < \frac{15 - \sqrt{105}}{4} < \frac{3}{2} < 6 < \frac{15 + \sqrt{105}}{4},$$

both (6.19) and (6.21) are fulfilled if and only if c satisfies (6.19). That is, both Inequalities (6.7) and (6.8) hold if and only if c is such that (6.19) is satisfied. Thus, if we choose $c = \frac{3}{2}$, then Result 1 leads to the following result:

The delay differential equation

$$x''(t) + \frac{8}{21(t+1)^5} [x(t-1)]^2 \operatorname{sgn} x(t-1) + \frac{1}{3(t+1)^3} x'(t) = 0$$

has at least one solution x on the interval $[0, \infty)$ such that

$$x(t) = \frac{5}{6}t + 1 + o(1) \quad \text{for } t \to \infty \tag{6.22}$$

and

$$x'(t) = \frac{5}{6} + o(1) \quad \text{for } t \to \infty; \tag{6.23}$$

in addition, this solution x satisfies

$$x(t) = x(0) \quad \text{for } -1 \leq t \leq 0, \tag{6.24}$$

$$\frac{5}{6}t+\frac{1}{2}\leq x(t)\leq\frac{5}{6}t+\frac{3}{2}\quad \textit{for every } t\geq 0 \tag{6.25}$$

and

$$\frac{1}{6}\leq x'(t)\leq\frac{3}{2}\quad \textit{for every } t\geq 0. \tag{6.26}$$

Example 2. Consider the delay differential equation (6.2) with $r=1$, $\alpha=2$, and

$$a(t)=\frac{16}{35(t+1)^5}\quad \text{for } t\geq 0.$$

Take $\xi=\frac{6}{5}$ and $\eta=1$. Inequality (6.9) is written

$$c^2\left[\int_0^1 t\frac{16}{35(t+1)^5}dt+\int_1^\infty t^3\frac{16}{35(t+1)^5}dt\right]\leq c-\frac{6}{5},$$

i.e.

$$\frac{2}{15}c^2\leq c-\frac{6}{5}. \tag{6.27}$$

We immediately see that (6.27) is satisfied if and only if (6.19) holds. That is, Inequality (6.9) holds true if and only if c satisfies (6.19). Choose $c=\frac{3}{2}$. Then, by applying Result 2, we arrive at the next result:

The delay differential equation

$$x''(t)+\frac{16}{35(t+1)^5}\left[x(t-1)\right]^2\operatorname{sgn}x(t-1)=0$$

has at least one solution x on the interval $[0,\infty)$ such that

$$x(t)=\frac{6}{5}t+1+o(1)\quad \textit{for } t\to\infty \tag{6.28}$$

and

$$x'(t)=\frac{6}{5}+o(1)\quad \textit{for } t\to\infty; \tag{6.29}$$

in addition, this solution x satisfies (6.24) and:

$$\frac{6}{5}t+\frac{7}{10}\leq x(t)\leq\frac{6}{5}t+\frac{13}{10}\quad \textit{for every } t\geq 0 \tag{6.30}$$

and

$$\frac{9}{10}\leq x'(t)\leq\frac{3}{2}\quad \textit{for every } t\geq 0. \tag{6.31}$$

Example 3. Let us consider the ordinary differential equation (6.3) with $\alpha=2$, $\beta=1$, and

$$a(t)=\frac{2}{9(t+1)^5}\quad \text{for } t\geq 0,\qquad b(t)=\frac{1}{3(t+1)^3}\quad \text{for } t\geq 0.$$

Let us take $\xi=\frac{5}{6}$ and $\eta=1$. Then, Inequality (6.10) becomes

$$c^2\int_0^\infty t\frac{2}{9(t+1)^3}dt+c\int_0^\infty t\frac{1}{3(t+1)^3}dt\leq c-1,$$

which leads to (6.18). Also, Inequality (6.11) is written as

$$c^2 \int_0^\infty \frac{2}{9(t+1)^3} dt + c \int_0^\infty \frac{1}{3(t+1)^3} dt \le c - \frac{5}{6},$$

which is equivalent to (6.20). As in Example 1, we see that both (6.18) and (6.20) are satisfied if and only if (6.19) holds. So, both Inequalities (6.10) and (6.11) hold if and only if c satisfies (6.19). Thus, by applying Result 3 with $c = \frac{3}{2}$, we are led to the following result:

The ordinary differential equation

$$x''(t) + \frac{2}{9(t+1)^5} [x(t)]^2 \operatorname{sgn} x(t) + \frac{1}{3(t+1)^3} x'(t) = 0$$

has at least one solution x on the interval $[0, \infty)$ such that (6.22) *and* (6.23) hold; *in addition, this solution x satisfies* (6.25) *and* (6.26).

Example 4. Let us consider the ordinary differential equation (6.4) with $\alpha = 2$ and

$$a(t) = \frac{4}{15(t+1)^5} \quad \text{for } t \ge 0.$$

Let us take $\xi = \frac{6}{5}$ and $\eta = 1$. In this case, Inequality (6.12) is written as follows

$$c^2 \int_0^\infty t \frac{4}{15(t+1)^3} dt \le c - \frac{6}{5},$$

which is equivalent to (6.27). But, (6.27) holds if and only if c satisfies (6.19). That is, Inequality (6.12) is fulfilled if and only if c is such that (6.19) holds. So, an application of Result 4 with $c = \frac{3}{2}$ leads to the next result:

The ordinary differential equation

$$x''(t) + \frac{4}{15(t+1)^5} [x(t)]^2 \operatorname{sgn} x(t) = 0$$

has at least one solution x on the interval $[0, \infty)$ such that (6.28) *and* (6.29) *hold; in addition, this solution x satisfies* (6.30) *and* (6.31).

Example 5. Consider the linear ordinary differential equation (6.5) with

$$a(t) = b(t) = \frac{1}{2(t+1)^4} \quad \text{for } t \ge 0.$$

We find

$$\int_0^\infty t \frac{1}{(t+1)^3} dt = \int_0^\infty \frac{1}{(t+1)^3} dt = \frac{1}{2}$$

and hence (6.13) is always satisfied. Now, take $\xi = \frac{5}{6}$ and $\eta = 1$. Inequality (6.14) becomes

$$c \left[\int_0^\infty t \frac{1}{2(t+1)^3} dt + \int_0^\infty t \frac{1}{2(t+1)^4} dt \right] \le c - 1,$$

i.e

$$\frac{1}{3} c \le c - 1 \quad \text{or} \quad c \ge \frac{3}{2}.$$

Moreover, Inequality (6.15) is written as

$$c \left[\int_0^\infty \frac{1}{2(t+1)^3} dt + \int_0^\infty \frac{1}{2(t+1)^4} dt \right] \le c - \frac{5}{6},$$

i.e

$$\frac{5}{12}c \leq c - \frac{5}{6} \quad \text{or} \quad c \geq \frac{10}{7}.$$

Thus, both Inequalities (6.14) and (6.15) are satisfied if and only if $c \geq \frac{3}{2}$. So, by applying Result 5 with $c = \frac{3}{2}$, we are immediately led to the following result:

The linear ordinary differential equation

$$x''(t) + \frac{1}{2(t+1)^4}x(t) + \frac{1}{2(t+1)^4}x'(t) = 0$$

has exactly one solution x on the interval $[0, \infty)$ satisfying

$$|x(0)| \leq \frac{3}{2}$$

and

$$|x'(t)| \leq \frac{3}{2} \quad \textit{for every } t \geq 0,$$

and such that (6.22) *and* (6.23) *hold; in addition, this unique solution x satisfies* (6.25) *and* (6.26).

Example 6. Consider the linear ordinary differential equation (6.6) with

$$a(t) = \frac{2}{5(t+1)^4} \quad \text{for } t \geq 0.$$

Since

$$\int_0^\infty t\frac{2}{5(t+1)^3}dt = \frac{1}{5},$$

we see that (6.16) is always satisfied. Now, take $\xi = \frac{6}{5}$ and $\eta = 1$. Inequality (6.17) is written as

$$c\int_0^\infty t\frac{2}{5(t+1)^3}dt \leq c - \frac{6}{5},$$

i.e

$$\frac{1}{5}c \leq c - \frac{6}{5} \quad \text{or} \quad c \geq \frac{3}{2}.$$

So, Inequality (6.17) holds true if and only if $c \geq \frac{3}{2}$. Hence, an application of Result 6 with $c = \frac{3}{2}$ gives the next result:

The linear ordinary differential equation

$$x''(t) + \frac{2}{5(t+1)^4}x(t) = 0$$

has exactly one solution x on the interval $[0, \infty)$ satisfying

$$|x(t)| \leq \frac{3}{2}(t+1) \quad \textit{for every } t \geq 0,$$

and such that (6.28) *and* (6.29) *hold; in addition, this unique solution x satisfies* (6.30) *and* (6.31).

Finally, we give an example related to our comment at the end of Section 2.

Example 7. Consider the linear ordinary differential equation

$$x''(t) + \frac{e^{-6t+1}}{t+1}x(t) = 0. \tag{6.32}$$

This equation is of the form (6.4) with $\alpha = 1$ and

$$a(t) = \frac{e^{-6t+1}}{t+1} \quad \text{for } t \geq 0.$$

Take $\xi = \frac{7}{10}$ and $\eta = \frac{9}{10}$, and choose $c = 1$. Then, Inequality (6.12) becomes

$$e\int_0^\infty te^{-6t}dt \leq \frac{1}{10}, \quad \text{i.e.} \quad \frac{e}{36} \leq \frac{1}{10}.$$

Thus, (6.12) holds true. Consequently, Result 4 guarantees the following:

The linear ordinary differential equation (6.32) *has at least one solution x on the interval $[0, \infty)$ such that*

$$x(t) = \frac{7}{10}t + \frac{9}{10} + o(1) \quad \textit{for } t \to \infty$$

and

$$x'(t) = \frac{7}{10} + o(1) \quad \textit{for } \ t \to \infty;$$

in addition, this solution x satisfies

$$\frac{7}{10}t + \frac{4}{5} \leq x(t) \leq \frac{7}{10}t + 1 \quad \textit{for every } t \geq 0$$

and

$$\frac{7}{10} - \frac{e}{6} \leq x'(t) \leq \frac{7}{10} + \frac{e}{6} \quad \textit{for every } t \geq 0.$$

Now, we observe that Equation (6.32) can also be obtained from (6.3) by taking $\alpha = \beta = 1$, and

$$a(t) = \frac{e^{-6t+1}}{t+1} \quad \text{for } t \geq 0, \qquad b(t) = 0 \quad \text{for } t \geq 0.$$

Again, we take $\xi = \frac{7}{10}$ and $\eta = \frac{9}{10}$, and we choose $c = 1$. Then, Inequality (6.11) is written

$$e\int_0^\infty e^{-6t}dt \leq \frac{3}{10}, \quad \text{i.e.} \quad \frac{e}{6} \leq \frac{3}{10}.$$

So, (6.11) fails to hold. Thus, Result 3 is not applicable. Consequently, the above result for the linear ordinary differential equation (6.32) cannot be obtained from Result 3.

7. SOME SUPPLEMENTARY RESULTS

The results of this section are formulated as two theorems (Theorems I and II) and two corollaries (Corollaries I and II). Corollaries I and II are immediate consequences of Theorems I and II, respectively. Theorem I and Corollary I concern the delay differential equation (E), while Theorem II and Corollary II are dealing with the delay differential equation (E_0). It must be noted that Theorem I and Corollary I can be applied, in particular, to the delay differential equation (E$'$) and, especially, to the ordinary differential equation (D); analogously, Theorem II and Corollary II are applicable to the particular case of the delay differential equation (E'_0) as well as to the special case of the ordinary differential equation (D_0). These applications are left to the reader.

Theorem I. *Assume that* (2.1) *holds, where* F *is a nonnegative real-valued function defined on* $[0,\infty)\times C([-r,0],[0,\infty))\times[0,\infty)$*, which satisfies the Continuity Condition* (C). *Suppose that* (B) *is satisfied.*

Let c *be a given positive real number such that* (3.1) *holds, where the function* γ *in* $C([-r,\infty),[0,\infty))$ *depends on* c *and is defined by* (2.4). *Then every solution* x *on the interval* $[0,\infty)$ *of the delay differential equation* (E) *with*

$$\max\left\{\max_{-r\le t\le 0}|x(t)|\,,\ \sup_{t\ge 0}|x'(t)|\right\}\le c \tag{7.1}$$

satisfies (2.5) *and* (2.6), *where the real constants* ξ *and* η *depend on the solution* x *and are defined as follows:*

$$\xi = x'(0) - \int_0^\infty f(t,x_t,x'(t))dt \tag{7.2}$$

and

$$\eta = x(0) + \int_0^\infty t f(t,x_t,x'(t))dt. \tag{7.3}$$

Corollary I. *Assume that* (2.1) *holds, where* F *is a nonnegative real-valued function defined on* $[0,\infty)\times C([-r,0],[0,\infty))\times[0,\infty)$*, which satisfies the Continuity Condition* (C). *Suppose that* (B) *is satisfied.*

Assume that, for any positive real number c, (3.1) *holds, where the function* γ *in* $C([-r,\infty),[0,\infty))$ *depends on* c *and is defined by* (2.4). *Then every solution* x *on the interval* $[0,\infty)$ *of the delay differential equation* (E) *with bounded derivative on* $[0,\infty)$ *satisfies* (2.5) and (2.6), *where the real constants* ξ *and* η *depend on the solution* x *and are defined by* (7.2) *and* (7.3), *respectively.*

Theorem II. *Assume that* (2.10) *holds, where* F_0 *is a nonnegative real-valued function defined on* $[0,\infty)\times C([-r,0],[0,\infty))$*, which satisfies the Continuity Condition* (C_0). *Suppose that* (B_0) *is satisfied.*

Let c *be a given positive real number such that* (4.1) *holds, where the function* γ *in* $C([-r,\infty),[0,\infty))$ *depends on* c *and is defined by* (2.4). *Then every solution* x *on the interval* $[0,\infty)$ *of the delay differential equation* (E_0) *with*

$$\max\left\{\max_{-r\le t\le 0}|x(t)|\,,\ \sup_{t\ge 0}\frac{|x(t)|}{t+1}\right\}\le c \tag{7.4}$$

satisfies (2.5) *and* (2.6), *where the real constants* ξ *and* η *depend on the solution* x *and are defined as follows:*

$$\xi = x'(0) - \int_0^\infty f_0(t,x_t)dt \tag{7.5}$$

and

$$\eta = x(0) + \int_0^\infty t f_0(t,x_t)dt. \tag{7.6}$$

Corollary II. *Assume that* (2.10) *holds, where* F_0 *is a nonnegative real-valued function defined on* $[0,\infty)\times C([-r,0],[0,\infty))$*, which satisfies the Continuity Condition* (C_0). *Suppose that* (B_0) *is satisfied.*

Assume that, for any positive real number c, (4.1) holds, where the function γ in $C([-r,\infty),[0,\infty))$ depends on c and is defined by (2.4). Then every solution x on the interval $[0,\infty)$ of the delay differential equation (E_0) with $x(t)=O(t)$ for $t\to\infty$ satisfies (2.5) and (2.6), where the real constants ξ and η depend on the solution x and are defined by (7.5) and (7.6), respectively.

Proof of Theorem I. Let x be a solution on the interval $[0,\infty)$ of the delay differential equation (E) such that (7.1) is satisfied. It follows from (7.1) that x satisfies (3.4) and (3.5). As in the proof of Proposition 1, we can arrive at (3.11), which guarantees that (7.2) and (7.3) define two real constants ξ and η, respectively, depending on the solution x.

Now, from (E) it follows that

$$x(t)=x(0)+tx'(0)-\int_0^t (t-s)f(s,x_s,x'(s))ds \quad \text{for } t\geq 0. \tag{7.7}$$

For every $t\geq 0$, we obtain

$$\begin{aligned}
&-\int_0^t (t-s)f(s,x_s,x'(s))ds\\
=\;& \int_0^t (t-s)d\left[\int_s^\infty f(\sigma,x_\sigma,x'(\sigma))d\sigma\right]\\
=\;& -t\int_0^\infty f(\sigma,x_\sigma,x'(\sigma))d\sigma+\int_0^t\left[\int_s^\infty f(\sigma,x_\sigma,x'(\sigma))d\sigma\right]ds\\
=\;& -t\int_0^\infty f(\sigma,x_\sigma,x'(\sigma))d\sigma+\int_0^\infty\left[\int_s^\infty f(\sigma,x_\sigma,x'(\sigma))d\sigma\right]ds\\
&-\int_t^\infty\left[\int_s^\infty f(\sigma,x_\sigma,x'(\sigma))d\sigma\right]ds\\
=\;& -t\int_0^\infty f(s,x_s,x'(s))ds+\int_0^\infty sf(s,x_s,x'(s))ds-\int_t^\infty (s-t)f(s,x_s,x'(s))ds.
\end{aligned}$$

Thus, (7.7) gives

$$\begin{aligned}
x(t)\;=\;& x(0)+tx'(0)-t\int_0^\infty f(s,x_s,x'(s))ds+\int_0^\infty sf(s,x_s,x'(s))ds\\
&-\int_t^\infty (s-t)f(s,x_s,x'(s))ds\\
=\;& \left[x'(0)-\int_0^\infty f(s,x_s,x'(s))ds\right]t+\left[x(0)+\int_0^\infty sf(s,x_s,x'(s))ds\right]\\
&-\int_t^\infty (s-t)f(s,x_s,x'(s))ds.
\end{aligned}$$

Hence, in view of (7.2) and (7.3), we have

$$x(t)=\xi t+\eta-\int_t^\infty (s-t)f(s,x_s,x'(s))ds \quad \text{for } t\geq 0. \tag{7.8}$$

But, (3.11) ensures that

$$\lim_{t\to\infty}\int_t^\infty (s-t)f(s,x_s,x'(s))ds=0.$$

So, it follows from (7.8) that the solution x satisfies (2.5). Furthermore, from (7.8) we obtain

$$x'(t) = \xi + \int_t^\infty f(s, x_s, x'(s))ds \quad \text{for } t \geq 0. \tag{7.9}$$

But, because of (3.11), it holds

$$\lim_{t\to\infty} \int_t^\infty f(s, x_s, x'(s))ds = 0.$$

Thus, (7.9) implies that x satisfies (2.6).

The proof of the theorem has been completed.

Proof of Theorem II. Let x be a solution on the interval $[0, \infty)$ of the delay differential equation (E_0), which satisfies (7.4). We immediately observe that (7.4) guarantees that the solution x is such that (3.4) and (3.6) hold. As in the proof of Proposition 2, we can conclude that (4.7) holds true and hence (7.5) and (7.6) define two real constants ξ and η, respectively, which depend on the solution x.

The rest of the proof of the theorem is similar with the corresponding part of the proof of Theorem I, and will be omitted.

Before closing this section and ending the paper, we note that the problem of giving sufficient conditions for every solution to be asymptotic at ∞ to a line (depending on the solution) has recently been investigated in [10], for second order nonlinear ordinary differential equations. For the more general case of n-th order ($n > 1$) nonlinear ordinary differential equations, conditions have been established in [17,18], which guarantee that every solution is asymptotic at ∞ to a real polynomial of degree at most $n - 1$ (depending on the solution).

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